

# Rules for the construction of symmetrical rhombic rosettes

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## 0. INTRODUCTION

A rhombic rosette is an edge-to-edge tiling of the regular convex polygon  $\{2n\}$  by  $\binom{n}{2}$  rhombs of  $\lfloor n/2 \rfloor$  different shapes. These  $\lfloor n/2 \rfloor$  rhombs, which we call the Standard Rhombic Inventory  $SRI_n$ , have smaller face angles  $k\pi/n$  ( $1 \leq k \leq \lfloor n/2 \rfloor$ );  $k$  is called the *principal index* of the rhomb, and  $n-k$  is called its *supplementary index* (since the larger face angle of the rhomb is  $(n-k)\pi/n$ ). We define each rhomb to have sides of unit length.

Rhombic rosettes have either

*dihedral* symmetry  $ds$ , where  $s$  is any odd divisor of  $n$ , or  
*cyclic* symmetry  $cs$ , where  $s$  is any *proper* odd divisor of  $n$ .

Fig. 0.1 shows rosettes of five different symmetry types for  $n=9$ ; Fig. 0.2 shows rosettes of four different symmetry types for  $n=10$ . I describe below a systematic procedure that facilitates the construction of both dihedral and cyclic rosettes ‘by hand’, *i.e.*, by arranging paper rhombs.

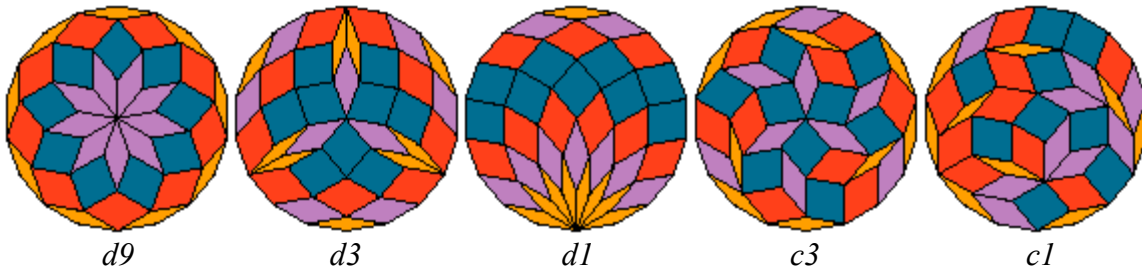


Fig. 0.1  
Rhombic rosettes for  $n=9$

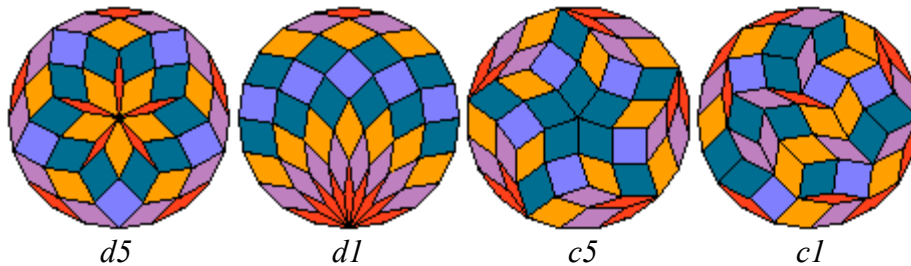


Fig. 0.2  
Rhombic rosettes for  $n=10$

# 1. DIHEDRAL ROSETTES

A dihedral rosette with symmetry  $ds$  contains  $s$  *diametral pendants*—linear strings of rhombs bounded by opposite boundary vertices. Each such pendant is symmetrical by reflection in its *axis*—a circumdiameter that is one of the  $s$  lines of reflection of the rosette. Figs. 1.1-1.2 show a  $d9$  rosette and a  $d5$  rosette, together with their diametral pendants and *diametral index strings* (defined below).

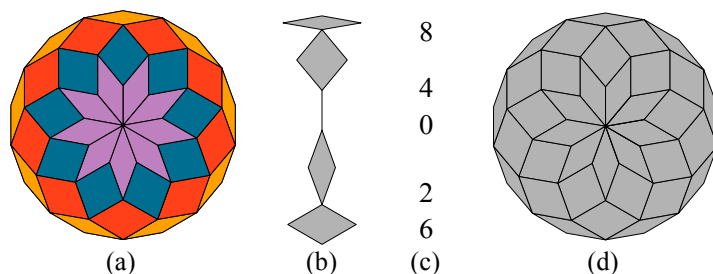


Fig. 1.1. (a)  $d9$  rosette; (b) one diametral pendant; (c) pendant index string; (d) the nine diametral pendants

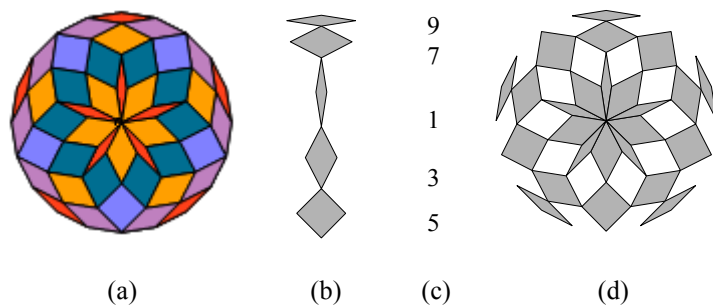


Fig. 1.2. (a)  $d5$  rosette; (b) one diametral pendant; (c) pendant index string; (d) the five diametral pendants

For each rhomb in a diametral pendant, vertices that lie on the pendant axis are called *axial vertices*, and the face-angle index at an axial vertex is called an *axial index*. If the axial vertices of a rhomb are even, the rhomb is called even; otherwise it is called odd. The sequence of  $\lfloor (n+1)/2 \rfloor$  axial indices for the  $\lfloor (n+1)/2 \rfloor$  rhombs of a diametral pendant is called the *diametral index string*. Examples of these strings are shown in Figs. 1.1c and 1.2c.

Every diametral pendant for  $s \geq 3$  is partitioned at the center of the rosette into two *radial pendants* of equal projected length but different composition. Hence the diametral index string is the union of two complementary substrings called *radial index strings*.

An efficient method of designing a dihedral rosette by hand is to construct the radial pendants first and then fill in the sectors between them. Except in the case of small  $n$ , however, it is not obvious which rhombs to assign to the two radial pendants and how to orient them. This note describes a systematic procedure that is conjectured to define, for all  $n \geq 3$ , **the composition of every possible pair of complementary radial pendants**. The *arrangement* of the rhombs in each radial pendant is left unspecified (but see the hint below).

A diametral pendant is composed of  $n/2$  **odd** rhombs if  $n$  is **even**, and of  $(n+1)/2$  **even** rhombs—including one *zero rhomb*, defined as a line segment of unit length—if  $n$  is **odd**.

I conjecture that for every partition of diametral rhombs generated by the procedure described below, there exists at least one dihedral rosette with radial pendants of composition defined by the partition. Although I have confirmed the truth of this conjecture in every example tested, I have not succeeded in constructing a general proof. Experience shows that once a pendant partition has been selected, it is not difficult to discover compatible arrangements of rhombs in the radial pendants, *i.e.*, arrangements that allow the rosette tiling to be completed. It may be possible to predict precisely which such sequences are compatible, but I do not know how to do this.

A hint about one kind of compatible arrangement of the rhombs in radial pendants is suggested by the dihedral rosettes in Figs. 0.1 and 0.2: *Let axial indices increase from the center outward.*

### Procedure for partitioning a diametral pendant into two radial pendants A and B

(a)  $n$  is even

$s$  is any odd divisor  $\geq 3$  of  $n$

(i) Construct a  $n/2 \times 2$  matrix of bins: two rows—A and B—and  $n/2$  columns, each headed by the name of one of the  $n/2$  axial indices  $1, 3, 5, \dots, n-3, n-1$ . The procedure will assign axial indices to the bins in a variety of different arrangements.

	1	3	5	.....	$n-5$	$n-3$	$n-1$
A	1						
B							

Axial indices assigned to row A are denoted as  $\bar{k}_i$  ( $1 \leq i \leq \alpha$ ); those assigned to row B are denoted as  $\underline{k}_j$  ( $1 \leq j \leq \beta$ ), where  $\alpha + \beta = n/2$ .

Let  $\bar{K} = \{\bar{k}_1, \bar{k}_2, \dots, \bar{k}_\alpha\}$  = the set of  $\alpha$  axial indices assigned to row A, and

$\underline{K} = \{\underline{k}_1, \underline{k}_2, \dots, \underline{k}_\beta\}$  = the set of  $\beta$  axial indices assigned to row B.

The solution set  $\bar{K}$  of entries in row A defines a radial index string, and the solution set  $\underline{K}$  in row B defines its complement. The union  $\bar{K} \cup \underline{K} =$  the set  $\{1, 3, 5, \dots, n-1\}$  of  $\alpha + \beta$  ( $= n/2$ ) axial indices in the diametral index string.

(ii) Define a “sum relation” to be an equation of type

$$\bar{k}_i + \underline{k}_j = \frac{2n}{s} \quad (1 \leq i \leq \alpha; 1 \leq j \leq \beta) \tag{1.1}$$

and a “difference relation” to be an equation of type

$$|\bar{k}_i - \underline{k}_j| = \frac{2n}{s} \quad (1 \leq i \leq \alpha; 1 \leq j \leq \beta). \tag{1.2}$$

(iii) Place axial index 1 in the leftmost bin of row A. Then assign each of the remaining indices either to row A or to row B, in every arrangement that contains  $\sigma$  sum relations and  $\delta$  difference relations, where

$$\sigma = \frac{n}{2s} \quad (1.3)$$

$$\delta = \frac{n}{2s}(s-2). \quad (1.4)$$

(b)  $n$  is odd

$s$  is any odd divisor  $\geq 3$  of  $n$

(i) Construct a  $(n+1)/2 \times 2$  matrix of bins: two rows—A and B—and  $(n+1)/2$  columns, each headed by the name of one of the axial indices  $0, 2, 4, \dots, n-3, n-1$ . The procedure will assign axial indices to the bins in a variety of different arrangements.

	0	2	4		-----		$n-5$	$n-3$	$n-1$
A	<b>0</b>				-----				
B					-----				

Axial indices assigned to row A are denoted as  $\bar{k}_i$  ( $1 \leq i \leq \alpha$ ); those assigned to row B are denoted as  $\underline{k}_j$  ( $1 \leq j \leq \beta$ ), where  $\alpha + \beta = n/2$ .

Let  $\bar{K} = \{\bar{k}_1, \bar{k}_2, \dots, \bar{k}_\alpha\}$  = the set of  $\alpha$  axial indices assigned to row A, and

$\underline{K} = \{\underline{k}_1, \underline{k}_2, \dots, \underline{k}_\beta\}$  = the set of  $\beta$  axial indices assigned to row B.

The solution set  $\bar{K}$  of entries in row A defines a radial index string, and the solution set  $\underline{K}$  in row B defines its complement. The union  $\bar{K} \cup \underline{K} =$  the set  $\{0, 2, 4, \dots, n-1\}$  of  $\alpha + \beta = (n+1)/2$  axial indices in the diametral index string.

(ii) Define a “sum relation” to be an equation of type

$$\bar{k}_i + \underline{k}_j = \frac{2n}{s} \quad (1 \leq i \leq \alpha; 1 \leq j \leq \beta) \quad (1.5)$$

and a “difference relation” an equation of type

$$|\bar{k}_i - \underline{k}_j| = \frac{2n}{s} \quad (1 \leq i \leq \alpha; 1 \leq j \leq \beta) \quad (1.6)$$

(iii) Place axial index 0 in the leftmost bin in row A. Then assign each of the remaining indices either to row A or to row B, in every arrangement that contains  $\sigma$  sum relations and  $\delta$  difference relations, where

$$\sigma = \frac{n}{2s} + \frac{1}{2} \quad (1.7)$$

$$\delta = \frac{n}{2s}(s-2) + \frac{1}{2}. \quad (1.8)$$

(Reminder: The projected length  $\lambda$  of a rhomb on the pendant axis is equal to  $2 \cos(\theta/2)$ , where  $\theta$  the face angle of a pendant rhomb at each of its axial vertices.)

Let  $s$  be any odd divisor  $\geq 3$  of  $n$ . Then every pair of complementary solution sets  $\bar{K}$  and  $\underline{K}$  satisfies the equation

$$\sum_{i=1}^{\alpha} \cos \bar{k}_i \frac{\pi}{2n} = \sum_{j=1}^{\beta} \cos \underline{k}_j \frac{\pi}{2n} \quad (1.9)$$

$$= r/2 \quad (1.10)$$

where  $r$ , the inradius of the rosette, is

$$r = \left(\frac{1}{2}\right) \csc \frac{\pi}{2n}. \quad (1.11)$$

### The number of solution sets

Let  $P(n,s)$  = the number of solution sets for a rosette of order  $n \geq 3$  and dihedral symmetry  $ds$  ( $s \geq 3$ ). Then

$$P(n,s) = 2^{\sigma-1} \quad (1.12)$$

#### Proof (even $n$ )

After the '1' is assigned (without loss of generality) to row A in the first column of bins, there remain  $2^{\sigma-1}$  distinct ways to assign indices in the next  $\sigma-1$  columns, *i.e.*, the columns for

indices 3, 5, ...,  $2\sigma-1$ .

$$\text{Let} \quad \omega = \frac{2n}{s} (= 4\sigma). \quad (1.13)$$

The assigning of indices to the next  $\sigma$  columns, *i.e.*, the columns for

indices  $2\sigma+1, 2\sigma+3, \dots, 4\sigma-3, 4\sigma-1$ ,

is dictated by the  $\sigma$  sum relations (Eq. 1.5):

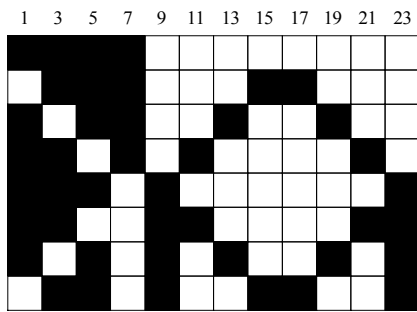
$$\begin{aligned} 1 &+ 4\sigma-1 = \omega \\ 3 &+ 4\sigma-3 = \omega \\ &\dots \\ 2\sigma-3 &+ 2\sigma+3 = \omega \\ 2\sigma-1 &+ 2\sigma+1 = \omega \end{aligned} \quad (1.14)$$

The assigning of indices to the last  $\frac{n}{2} - 2\sigma = \delta$  columns, *i.e.*, the columns for indices  $4\sigma + 1, 4\sigma + 3, \dots, n - 1,$

is dictated by the  $\delta$  difference relations (*cf.* Eq. 1.6). Hence the  $2^{\sigma-1}$  distinct ways of assigning indices to the first  $\sigma$  columns exhaust all possibilities.  $\square$

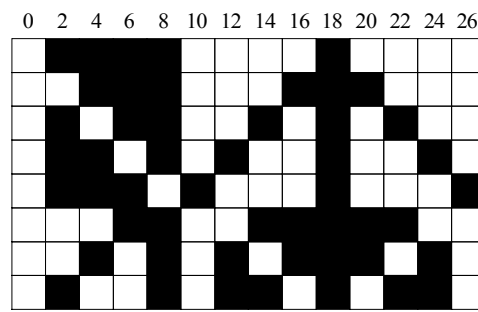
(The proof for odd  $n$  is left to the reader.)

The composition of the pairs of complementary radial pendants is displayed in graphical form in Figs. 1.3-1.5 for  $(n,s) = (24,3), (27,3),$  and  $(30,3)$ . These diagrams illustrate how complete solution sets are derived from the first  $\sigma$  columns by combining the elementary symmetry operations (a) translation, (b) reflection, and (c) inversion (*i.e.*, reflection followed by exchange of rows).



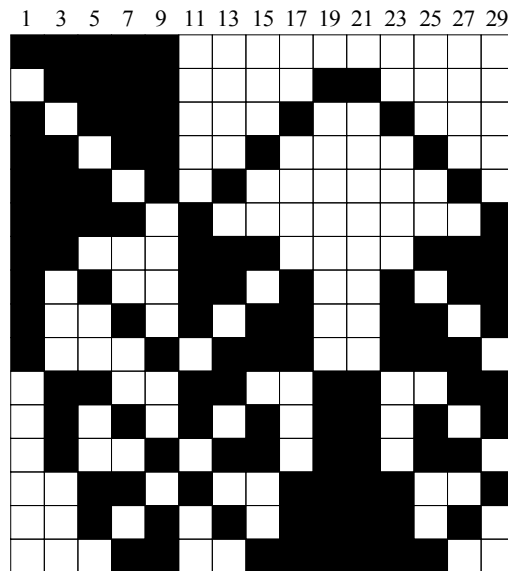
$(n,s) = (24,3)$

Fig. 1.3



$(n,s) = (27,3)$

Fig. 1.4



$(n,s) = (30,3)$

Fig. 1.5

Appendix 1 lists every possible solution set for  $3 \leq n \leq 30$ , for all allowed values of  $s$ .

## 2. CYCLIC ROSETTES

The sequence of rhombs in every ladder is described by a *ladder string*—an ordered list of the  $n-1$  *ladder indices*  $k$  of the face angles  $k\pi/n$  ( $k \in [1, n-1]$ ) at the lower left corners of the ladder rhombs.

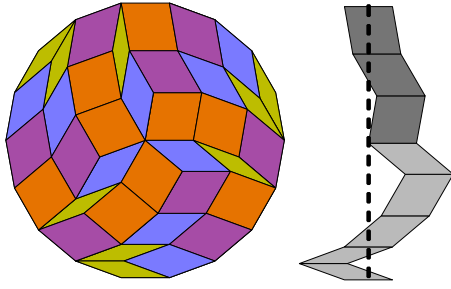
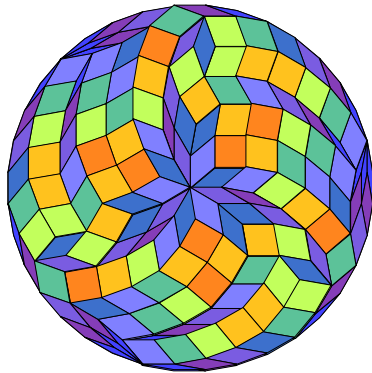


Fig. 2.1

A  $c3$  rosette ( $n=9$ ) and a balanced ladder



Figs. 2.1 and 2.2 show two examples of a  $c3$  rosette. The indiameter perpendicular to each ladder's *rungs* (edges common to adjacent rhombs in the ladder) is called the *axis* of the ladder (*cf.* dashed line in Fig. 2.1). Six of the nine ladders of this  $c3$  rosette are *balanced*, *i.e.*, they are partitioned at the center of the rosette into complementary *half-ladders*, each of which has projected length on the ladder axis equal to the inradius  $r = 1 / (2 \tan(\pi / 2n))$ . In every rosette with cyclic symmetry  $cs$  ( $s$ =odd integer  $\geq 3$ ) there are at least  $s$  balanced ladders.

Designing a cyclic rosette by hand begins with the *balanced ladder algorithm*—a recipe for (I) deriving every possible *composition* of two complementary half-ladders and (II) specifying the *orientation* of all the non-square rhombs (*i.e.*, whether they ‘lean’ to the left or to the right). After the  $n-1$  rhombs are arranged in a particular sequence,  $s$  congruent replicas of the ladder are placed inside the tiling arena. Finally the remaining rhombs in  $SRI_n$  are used to tile the sectors between the  $s$  ladders.

The string for the balanced ladder in Fig 2.1 is  $\{8,1,2,3,7 \mid 4,6,5\}$ . The strings for the balanced ladders at the bottom of Fig. 2.2, beginning with the darker half-ladder, are:

- $\{17,13,1,2,3,5,6,10,14,15 \mid 4,9,8,12,7,11,16\}$  (left)
- $\{1,2,6,9,10,11,13,14 \mid 5,4,8,7,3,12,15,16,17\}$  (middle)
- $\{1,17,16,4,5,6,8,9,13 \mid 3,2,7,12,11,15,10,14\}$  (right)

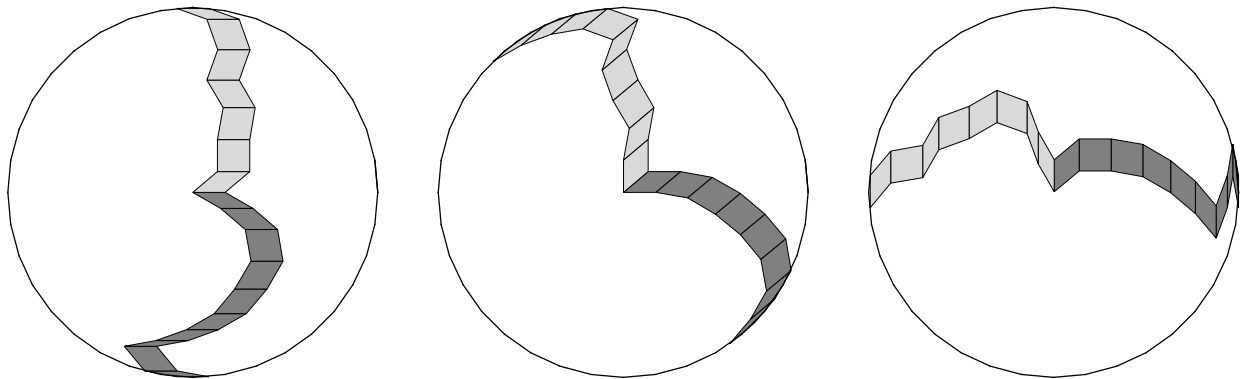


Fig. 2.2

Top: A  $c3$  rosette ( $n=18$ )

Bottom: Three *balanced ladders*, partitioned at the center of the rosette into two half-ladders of equal projected length

## The balanced ladder algorithm

### I. Assigning the $n - 1$ ladder rhombs to two complementary half-ladders

Let  $n =$  a composite positive integer  $\geq 6$  and  $s =$  any proper odd divisor  $\geq 3$  of  $n$ :

$$n = s q. \tag{2.1}$$

(i) First construct the **skeletal sequence**, composed of  $s$  terms with increment  $q$ .

(a)  $s \equiv 1 \pmod{4\mu}$  ( $\mu = 1, 2, \dots$ )

The skeletal sequence is:

$$\begin{array}{c} \overline{0} \quad \underline{q} \quad \overline{2q} \quad \underline{3q} \quad \dots \quad \overline{\frac{1}{2}(s-1)q} \quad | \quad \overline{q} \quad \underline{2q} \quad \overline{3q} \quad \dots \quad \overline{\frac{1}{2}(s-1)q} \\ \underbrace{\hspace{10em}}_{\frac{1}{2}(s+1) \text{ terms}} \quad | \quad \underbrace{\hspace{10em}}_{\frac{1}{2}(s-1) \text{ terms}} \\ \underbrace{\hspace{15em}}_{(s \text{ terms altogether})} \end{array} \tag{2.2}$$

(b)  $s \equiv -1 \pmod{4\mu}$  ( $\mu = 1, 2, \dots$ )

The skeletal sequence is:

$$\begin{array}{c} \overline{0} \quad \underline{q} \quad \overline{2q} \quad \underline{3q} \quad \dots \quad \overline{\frac{1}{2}(s-1)q} \quad | \quad \overline{q} \quad \underline{2q} \quad \overline{3q} \quad \dots \quad \overline{\frac{1}{2}(s-1)q} \\ \underbrace{\hspace{10em}}_{\frac{1}{2}(s+1) \text{ terms}} \quad | \quad \underbrace{\hspace{10em}}_{\frac{1}{2}(s-1) \text{ terms}} \\ \underbrace{\hspace{15em}}_{(s \text{ terms altogether})} \end{array} \tag{2.3}$$

Upper and lower bars are assigned consecutively to the terms on each side of (2.2) and (2.3): Place a bar **above** term 1, **below** term 2, **above** term 3, ... (etc.).

An **upper bar** means **add** 1 to the term; a **lower bar** means **subtract** 1 from the term.  
(Repeat this procedure, so long as no term becomes greater than  $n / 2$ .)

For example, for  $(n, s, q) = (18, 9, 2)$ , the skeletal sequence is  $\overline{0} \quad \underline{2} \quad \overline{4} \quad \underline{6} \quad \overline{8} \quad | \quad \overline{2} \quad \underline{4} \quad \overline{6} \quad \underline{8}$ .

Applying the **bar rule** yields  $\overline{1} \quad \underline{1} \quad \overline{5} \quad \underline{5} \quad \overline{9} \quad | \quad \overline{3} \quad \underline{3} \quad \overline{7} \quad \underline{7}$ , which is called a **basic exchange relation**.

For  $(n, s, q) = (18, 3, 6)$ , the skeletal sequence is  $\overline{0} \quad \underline{6} \quad | \quad \overline{6}$ . Applying the bar rule three times in succession (while keeping all the bars in the same positions) yields three basic exchange relations

$$\overline{1} \quad \underline{5} \quad | \quad \overline{7}, \quad \overline{2} \quad \underline{4} \quad | \quad \overline{8}, \quad \text{and} \quad \overline{3} \quad \underline{3} \quad | \quad \overline{9}.$$

A *basic exchange relation* describes the composition of two *matched ribbons* of rhombs, *i.e.*, ribbons of equal projected length.  
It enables the transfer of rhombs between matched half-ladders.



The basic exchange relation  $\bar{1} \bar{1} \bar{5} \bar{5} \bar{9} \mid \bar{3} \bar{3} \bar{7} \bar{7}$  for  $(n, s, q) = (18, 9, 2)$ , is equivalent to

$$\sin(\pi / 18) + \sin(\pi / 18) + \sin(5\pi / 18) + \sin(5\pi / 18) + \sin(9\pi / 18) = \sin(3\pi / 18) + \sin(3\pi / 18) + \sin(7\pi / 18) + \sin(7\pi / 18), \text{ or}$$

$$2\sin(\pi / 18) + 2\sin(5\pi / 18) + \sin(9\pi / 18) = 2\sin(3\pi / 18) + 2\sin(7\pi / 18). \quad (2.4)$$

The *matched ribbons* of rhombs in Fig. 2.3 illustrate this exchange relation.

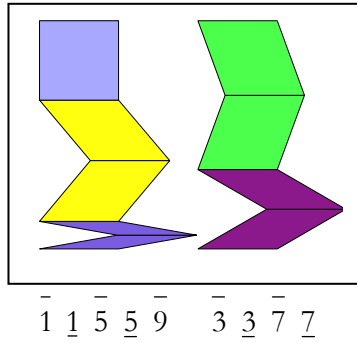


Fig. 2.3

The basic exchange relations  $\bar{1} \bar{5} \mid \bar{7}$ ,  $\bar{2} \bar{4} \mid \bar{8}$ , and  $\bar{3} \bar{3} \mid \bar{9}$  for  $(n, s, q) = (18, 3, 6)$  are equivalent to

$$\sin(\pi / 18) + \sin(5\pi / 18) = \sin(7\pi / 18), \quad (2.5a)$$

$$\sin(\pi / 9) + \sin(2\pi / 9) = \sin(4\pi / 9), \text{ and} \quad (2.5b)$$

$$2\sin(\pi / 6) = \sin(\pi / 2), \text{ respectively.} \quad (2.5c)$$

The matched ribbons of rhombs in Fig. 2.4 illustrate the basic exchange relations (2.5a), (2.5b), and (2.5c).

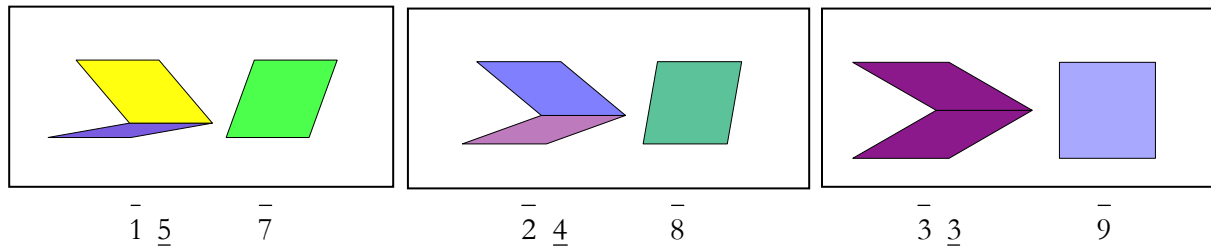


Fig. 2.4

For  $(n, s, q) = (9, 3, 3)$ , there is only one exchange relation:  $\bar{1} \bar{2} \mid \bar{4}$ , which is equivalent to the exchange relation  $\bar{2} \bar{4} \mid \bar{8}$  for  $(n, s, q) = (18, 3, 6)$  (cf. Fig 2.4).

The result of applying two or more *basic* exchange relations in succession is equivalent to applying a *composite* exchange relation.

For example,  $\bar{1} \bar{5} \mid \bar{7} + \bar{2} \bar{4} \mid \bar{8} \equiv \bar{1} \bar{5} \bar{2} \bar{4} \mid \bar{7} \bar{8}$

and  $\bar{1} \bar{5} \mid \bar{7} + \bar{8} \mid \bar{2} \bar{4} \equiv \bar{1} \bar{5} \bar{8} \mid \bar{7} \bar{2} \bar{4}.$

- (ii) Construct a *standard pair of **matched** half-ladders* if  $n$  is odd, and a *standard pair of **truncated** half-ladders* if  $n$  is even.

Then apply one or more basic exchange relations, thereby producing matched half-ladder strings.

For **odd**  $n$ , each of the two *matched* half-ladder strings of the *standard pair* in (2.7) contains the index for every shape of rhomb in  $SRI_n$ :

$$\{1, 2, \dots, (n-1)/2\} | \{1, 2, \dots, (n-1)/2\}. \quad (2.7)$$

For **even**  $n$ , producing a *standard pair of matched* half-ladder strings requires three steps. We illustrate with an example. Consider  $(n, s, q) = (18, 9, 2)$ .

- (a) First construct a pair of identical *truncated* strings, each containing the index for every rhomb *except the square*:

$$\{1, 2, \dots, 8\} | \{1, 2, \dots, 8\}. \quad (2.8)$$

- (b) Apply the basic exchange relation ERS that contains the square,  $\bar{1} \ \underline{1} \ \bar{5} \ \underline{5} \ \bar{9} \ | \ \bar{3} \ \underline{3} \ \bar{7} \ \underline{7}$ , to each string in (2.8). This means adding the rhombs on the left side of ERS to the left side of (2.8) and adding the rhombs on the right side of ERS to the right side of (2.8):

$$\{1^3, 2, 3, 4, 5^3, 6, 7, 8, 9\} | \{1, 2, 3^3, 4, 5, 6, 7^3, 8\}^1$$

- (c) Because the number of non-square rhombs in a ladder is limited to two, **reduce** the half-ladder strings in (2.9) by removing all extra indices:

$$\{1^2, 2, 4, 5^2, 6, 8, 9\} | \{2, 3^2, 4, 6, 7^2, 8\} \quad (2.10)$$

(2.10) is the required *standard pair of matched* half-ladder strings for  $(n, s, q) = (18, 9, 2)$ .

As a second example, consider  $(n, s, q) = (18, 3, 6)$  (cf. Fig. 2.2). Because  $n$  is even, steps (ii-a) and (ii-b) require that the exchange relation  $\bar{3} \ \underline{3} \ | \ \bar{9}$  be applied to the identical truncated strings  $\{1, 2, \dots, 8\} | \{1, 2, \dots, 8\}$  in (2.8), yielding

$$\{1, 2, 3^3, 4, 5, 6, 7, 8\} | \{1, 2, 3, 4, 5, 6, 7, 8, 9\}. \quad (2.11)$$

After reduction, (2.11) is transformed into the standard pair

$$\{1, 2, 3^2, 4, 5, 6, 7, 8\} | \{1, 2, 4, 5, 6, 7, 8, 9\}. \quad (2.12)$$

If the exchange relation  $\bar{1} \ \underline{5} \ | \ \bar{7}$  were applied to (2.12), the result would be

$$\{1^2, 2, 3^2, 4, 5^2, 6, 8\} | \{2, 4, 6, 7^2, 8, 9\}, \quad (2.13)$$

which defines the complementary half-ladders shown at bottom left in Fig. 2.2.

Every possible balanced ladder composition can be obtained by repeated application of basic exchange relations. If there are altogether  $m$  basic exchange relations, they can be combined to produce a total of  $(3^m - 1) / 2$  exchange relations (including both basic and composite relations).

<sup>1</sup> The notation  $index^j$  above means that there are  $j$  instances of the rhomb with index  $index$ .

## II. Specifying the orientation of the non-square rhombs in two complementary half-ladders

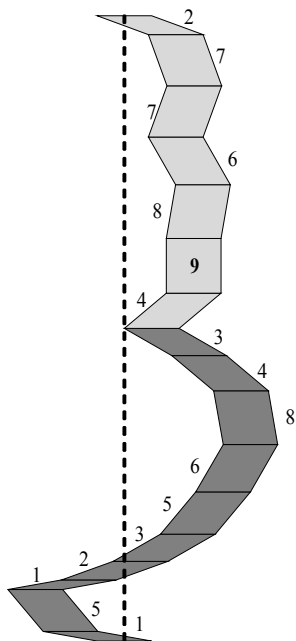


Fig. 2.5  
A balanced ladder for  $n=18$

Part I of the balanced ladder algorithm explained how to choose sets of rhombs that form connected ribbons that have the overall length required for a half-ladder. Because a circle inscribed in a regular polygon is tangent to the polygon at the midpoint of each polygon edge, in a balanced ladder there is a *lateral offset*—equal to one-half of the rhomb edge length—between the end rung of each half-ladder and the center of the rosette (*cf.* Fig. 2.5). Since the rhombs of each identical pair in a half-ladder are oppositely oriented (they are related by a glide reflection), they make no contribution to this lateral offset. It is the relative orientations of all the *singleton* rhombs in the half-ladder that accounts for the offset.

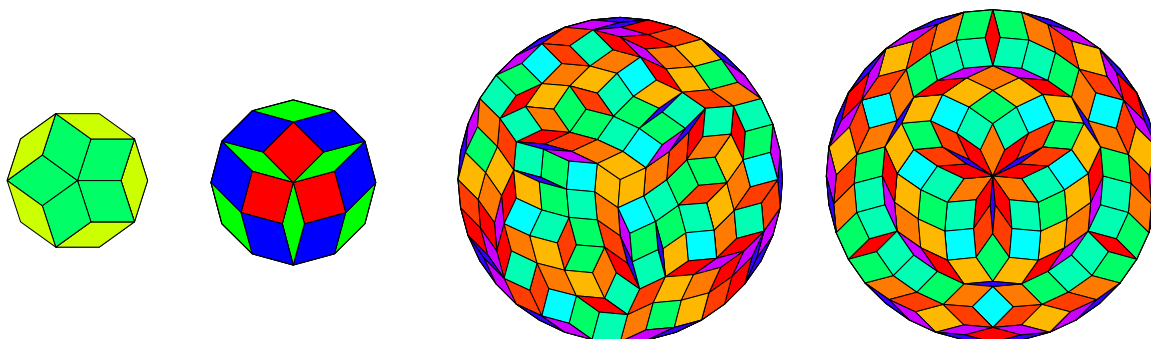
A simple procedure that specifies the orientations of the singleton rhombs is the *zigzag rule* (described below). The zigzag rule is a special case of a more general procedure called ‘*p*-fold *q*-chains’, which was discovered by closely examining the balanced ladders in a variety of computer-generated rosettes. In 1980, Andy Odlyzko (personal communication) proved the validity of the *p*-fold *q*-chains rule. A description of the rule, including Odlyzko’s proof, will be published on the author’s website, schoengeometry.com.)

### The zigzag rule:

1. Assign a *rank* from 1 to  $g$  to each of the  $g$  singletons in the half-ladder, in order of size.
2. Let the rhombs of even rank lean left and those of odd rank lean right (or *vice versa*).

In each of the half-ladders of Fig. 2.5, there are four singletons, with principal indices 2, 4, 6, 8. It is apparent that their orientations are consistent with the zigzag rule.

This summary is an edited version of the author’s report *Rhombic Rosettes*, issued in 1979 by the Dept. of Design, Southern Illinois University/Carbondale.





$n=10$

$s=5$      $P(10,5)=1$      $(\sigma, \delta, \omega) = (1,3,4)$

(1)	1	7	9
	3	5	

---

$n=11$

$s=11$      $P(11,11)=1$      $(\sigma, \delta, \omega) = (1,5,2)$

(1)	0	4	8
	2	6	10

---

$n=12$

$s=3$      $P(12,3)=2$      $(\sigma, \delta, \omega) = (2,2,8)$

(1)	1	3			
		5	7	9	11

---

(2)	1	5	11
	3	7	9

---

$n=13$

$s=13$      $P(13,13)=1$      $(\sigma, \delta, \omega) = (1,6,2)$

(1)	0	4	8	12
	2	6	10	

---

$n=14$

$s=7$      $P(14,7)=1$      $(\sigma, \delta, \omega) = (1,5,4)$

(1)	1	7	9	
	3	5	11	13

---

$n=15$

$s=3$      $P(15,3)=4$      $(\sigma, \delta, \omega) = (3,3,10)$

(1)	0	2	4			
		6	8	10	12	14

---

$n=15$  (cont.)

(2)                    0 2    6            14  
                              4    8 10 12

---

(3)                    0            6 8            12 14  
                              2 4            10

---

(4)                    0            4    8            12  
                              2            6    10    14

---

$s=5$      $P(15,5)=2$      $(\sigma, \delta, \omega) = (2,5,6)$

(1)                    0 2            10 12 14  
                              4 6 8

---

(2)                    0            4    8            12  
                              2            6    10    14

---

$s=15$      $P(15,15)=1$      $(\sigma, \delta, \omega) = (1,7,2)$

(1)                    0            4    8            12  
                              2            6    10    14

---

$n=17$

$s=17$      $P(17,17)=2$      $(\sigma, \delta, \omega) = (1,8,2)$

(1)                    0            4    8            12    16  
                              2            6    10    14

---

$n=18$

$s=3$      $P(18,3)=4$      $(\sigma, \delta, \omega) = (3,3,12)$

(1)                    1 3 5  
                                          7 9 11 13 15 17

---

(2)                    1 3            7            17  
                              5            9 11 13 15

---

$n=18$  (cont.)

(3)                                    1    5    9                    15  
                                          3       7            11 13       17

---

(4)                                    1            7 9                    15 17  
                                          3 5                    11 13

---

$s=9$      $P(18,9)=1$      $(\sigma, \delta, \omega) = (1,7,4)$

                                          1            7 9                    15 17  
                                          3 5                    11 13

---

$n=19$

$s=19$      $P(19,19)=1$      $(\sigma, \delta, \omega) = (1,9,2)$

(1)                                    0    4    8    12    16  
                                          2       6       10    14    18

---

$n=20$

$s=5$      $P(20,5)=2$      $(\sigma, \delta, \omega) = (2,6,8)$

(1)                                    1    5            11    15 17  
                                          3       7 9            13       19

---

(2)                                    1 3                    13 15 17 19  
                                          5 7 9 11

---

$n=21$

$s=3$      $P(21,3)=8$      $(\sigma, \delta, \omega) = (4,4,14)$

(1)                                    0 2 4 6  
                                                                  8 10 12 14 16 18 20

---

(2)                                    0 2 4            8                    20  
                                                          6            10 12 14 16 18

---

(3)                                    0 2            6    10                    18  
                                          4       8            12 14 16       20

---

$n=21$  (cont.)

(4)                    0    4    8    12   16   20  
                              2    6    10   14   18

---

(5)                    0 2        8 10        18 20  
                              4 6        12 14 16

---

(6)                    0 2 4 6  
                                          8 10 12 14 16 18 20

---

(7)                    0        6    10 12    16 18  
                  2 4        8        14        20

---

(8)                    0        8 10 12    16 18 20  
                  2 4 6        14

---

$s=7$      $P(21,7)=2$      $(\sigma, \delta, \omega) = (2, 8, 6)$

(1)                    0 2        10 12 14  
                              4 6 8        16 18 20

---

(2)                    0    4    8    12   16   20  
                  2    6    10   14   18

---

$s=21$      $P(21,21)=1$      $(\sigma, \delta, \omega) = (1, 10, 2)$

(1)                    0    4    8    12   16   20  
                  2    6    10   14   18

---

$n=22$

$s=11$      $P(22,11)=1$      $(\sigma, \delta, \omega) = (1, 9, 4)$

(1)                    1        7 9        15 17  
                  3 5        11 13        19 21

---



$s=23$   $P(23,23)=1$   $(\sigma, \delta, \omega) = (1, 11, 2)$

$n=23$

(1)                    0    4    8    12   16   20  
                          2    6    10   14   18   22

---

$s=3$   $P(24,3)=1$   $(\sigma, \delta, \omega) = (4, 4, 16)$

$n=24$

(1)                    1 3 5 7  
                                          9 11 13 15 17 19 21 23

---

(2)                    1 3 5                    23  
                                  7 9 11 13 15 17 19 21

---

(3)                    1 3                    7 11                    21  
                                  5 9                    13 15 17 19                    23

---

(4)                    1                    5 7                    13                    19  
                                  3                    9 11                    15 17                    21 23

---

(5)                    1 3                    9 11                    21 23  
                                  5 7                    13 15 17 19

---

(6)                    1                    5 7 9                    13 17 21  
                                  3                    11 15 19                    23

---

(7)                    1                    7 11 13                    19 21  
                                  3 5 9                    15 17                    23

---

(8)                    1                    9 11 13                    19 21 23  
                                  3 5 7                    15 17

---

$n=25$

$s=5$      $P(25,5)=4$      $(\sigma, \delta, \omega) = (3, 8, 10)$

(1)                    0 2 4                    16 18 20 22 24  
                                  6 8 10 12 14

---

(2)                    0 2        6                    14        18 20 22  
                                  4        8 10 12        16                    24

---

(3)                    0        4        8        12        16        20        24  
                                  2        6        10        14        18        22

---

(4)                    0            6 8        12 14            20  
                                  2 4            10            16 18        22 24

---

$n=26$

$s=13$      $P(26,13)=1$      $(\sigma, \delta, \omega) = (1, 11, 4)$

(1)                    1            7 9                    15 17            23 25  
                                  3 5                    11 13            19 21

---

$n=27$

$s=3$      $P(27,3)=16$      $(\sigma, \delta, \omega) = (5, 5, 18)$

(1)                    0 2 4 6 8                    10 12 14 16 18 20 22 24 26

---

(2)                    0 2 4 6        10                    26  
                                  8        12 14 16 18 20 22 24

---

(3)                    0 2 4        8        12                    24  
                                  6        10        14 16 18 20 22        26

---

(4)                    0 2        6 8                    14                    22  
                                  4                    10 12        16 18 20        24 26

---

(5)                    0        4 6 8                    16        20  
                                  2                    10 12 14        18        22 24 26

$n=27$  (cont.)

---

(6)            0 2 4            10 12            24 26  
                          6 8            14 16 18 20 22

---

(7)            0 2            6 8            10 14            22 26  
                          4            12 16 18 20            24

---

(8)            0            4 6            10            16 20            26  
                          2            8 12 14            18 22 24

---

(9)            0 2            8            12 14            22 24  
                          4 6            10            16 18 20            26

---

(10)           0            4 8            12 16 20 24  
                          2            6 10 14 18 22 26

---

(11)           0            6 8            14 16 20 22  
                          2 4            10 12            18 24 26

---

(12)           0 2            10 12 14            22 24 26  
                          4 6 8            16 18 20

---

(13)           0            4            10 12 16 20 24 26  
                          2            6 8            14 18 22

---

(14)           0            6 8            10 14 16 20 22 26  
                          2 4            12            18 24

---

(15)           0            8            12 14 16 20 22 24  
                          2 4 6            10            18 26

---

(16)           0            10 12 14 16 20 22 24 26  
                          2 4 6 8            18

---

n=27 (cont.)

s=9  $P(27,9)=2$   $(\sigma, \delta, \omega) = (2, 11, 6)$

(1) 0 2 4 6 8 10 12 14 16 18 20 22 24 26

---

(2) 0 4 8 12 16 20 24  
2 6 10 14 18 22 26

---

s=9  $P(27,27)=1$   $(\sigma, \delta, \omega) = (1, 13, 2)$

(1) 0 4 8 12 16 20 24  
2 6 10 14 18 22 26

---

s=7  $P(28,7)=2$   $(\sigma, \delta, \omega) = (2, 10, 8)$

(1) 1 3 5 7 9 11 13 15 17 19 21 23 25 27

---

(2) 1 3 5 7 9 11 13 15 17 19 21 23 25 27

---

n=29

s=7  $P(29,29)=1$   $(\sigma, \delta, \omega) = (1, 14, 2)$

(1) 0 4 8 12 16 20 24 28  
2 6 10 14 18 22 26

---

n=30

s=3  $P(30,3)=16$   $(\sigma, \delta, \omega) = (5, 5, 20)$

(1) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(2) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(3) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

$n=30$  (cont.)

(4) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(5) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(6) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(7) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(8) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(9) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(10) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(11) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(12) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(13) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(14) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(15) 1 3 5 7 9 11 13 15 17 19 21 23 25 27 29

---

(16)            1                    11 13 15 17            23 25 27 29  
                  3 5 7 9                    19 21

---

$s=5$      $P(30,5)=4$      $(\sigma, \delta, \omega) = (3,9,12)$

(1)            1 3 5                    19 21 23 25 27 29  
                              7 9 11 13 15 17

---

(2)            1 3        7                    17    21 23 25 27  
                  5        9 11 13 15    19                    29

---

(3)            1        5        9                    15 17 19    23 25    29  
                  3        7        11 13                    21            27

---

(4)            1            7 9                    15 17            23 25  
                  3 5                    11 13            19 21            27 29

---

$s=15$      $P(30,15)=1$      $(\sigma, \delta, \omega) = (1,13,4)$

(1)            1            7 9                    15 17            23 25  
                  3 5                    11 13            19 21            27 29

---