

# Rhombic tilings of $(n, k)$ -Ovals, $(n, k, \lambda)$ -cyclic difference sets, and related topics

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## ABSTRACT

Each fixed integer  $n$  has associated with it  $\lfloor \frac{n}{2} \rfloor$  rhombs:  $\rho_1, \rho_2, \dots, \rho_{\lfloor \frac{n}{2} \rfloor}$ , where, for each  $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$ , rhomb  $\rho_h$  is a parallelogram with all sides of unit length and with smaller face angle equal to  $h \times \frac{\pi}{n}$  radians.

An Oval is a centro-symmetric convex polygon all of whose sides are of unit length, and each of whose turning angles equals  $\ell \times \frac{\pi}{n}$  for some positive integer  $\ell$ . A  $(n, k)$ -Oval is an Oval with  $2k$  sides tiled with rhombs  $\rho_1, \rho_2, \dots, \rho_{\lfloor \frac{n}{2} \rfloor}$ ; it is defined by its Turning Angle Index Sequence, a  $k$ -composition of  $n$ . For any fixed pair  $(n, k)$  we count and generate all  $(n, k)$ -Ovals up to translations and rotations, and, using multipliers, we count and generate all  $(n, k)$ -Ovals up to congruency. For odd  $n$  if a  $(n, k)$ -Oval contains a fixed number  $\lambda$  of each type of rhomb  $\rho_1, \rho_2, \dots, \rho_{\lfloor \frac{n}{2} \rfloor}$  then it is called a magic  $(n, k, \lambda)$ -Oval. We prove that a magic  $(n, k, \lambda)$ -Oval is equivalent to a  $(n, k, \lambda)$ -Cyclic Difference Set. For even  $n$  we prove a similar result. Using tables of Cyclic Difference Sets we find all magic  $(n, k, \lambda)$ -Ovals up to congruency for  $n \leq 40$ .

Many related topics including lists of  $(n, k)$ -Ovals, partitions of the regular  $2n$ -gon into Ovals, Cyclic Difference Families, partitions of triangle numbers,  $u$ -equivalence of  $(n, k)$ -Ovals, etc., are also considered.

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## 1. Introduction

A  $(n, k)$ -Oval is a centro-symmetric convex polygon all of whose sides are of unit length, and which is tiled by rhombs; see p. 141 of Ball and Coxeter [1] and Section 3.1 of Schoen [8]. In this paper we investigate  $(n, k)$ -Ovals; it appears that this is the first significant piece of research concerning  $(n, k)$ -Ovals to be published in the mathematical literature. A preliminary version of some of this research first appeared in Schoen [8]. See Fig. 1 for an example of a  $(15, 6)$ -Oval.

In Section 2 of this paper we define a  $(n, k)$ -Oval using its Turning Angle Index Sequence (TAIS); we count all  $(n, k)$ -Ovals equivalent up to translations and rotations. We introduce the concept of a multiplier for a  $(n, k)$ -Oval and show how to generate all  $(n, k)$ -Ovals using multipliers.

In Section 3 we show the geometrical meaning of multiplier  $-1$  for a  $(n, k)$ -Oval. We count those  $(n, k)$ -Ovals with multiplier  $-1$ , and those without multiplier  $-1$ . We define congruency for  $(n, k)$ -Ovals and count  $(n, k)$ -Ovals up to congruency.

In Section 4 we define the Rhombic Inventory Vector (RIV) of a  $(n, k)$ -Oval. This vector contains the number of each type of rhomb that a  $(n, k)$ -Oval contains. For each  $2 \leq n \leq 10$  we list all  $(n, k)$ -Ovals up to congruency, and compute their RIVs.

In Section 5 we study magic  $(n, k, \lambda)$ -Ovals. For odd  $n$  a magic  $(n, k, \lambda)$ -Oval contains a fixed number  $\lambda \geq 1$  of each type of rhomb  $\rho_1, \rho_2, \dots, \rho_{\lfloor \frac{n}{2} \rfloor}$ ; there is a similar definition for even  $n$ . We prove that a magic  $(n, k, \lambda)$ -Oval is equivalent

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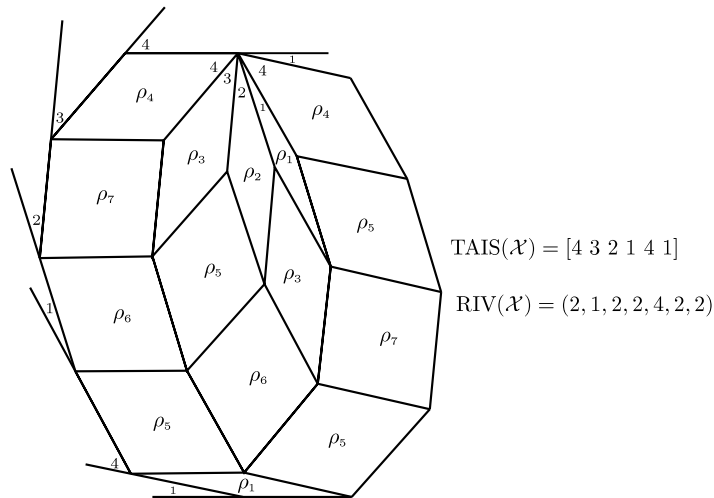


Fig. 1. A (15, 6)-Oval,  $\mathcal{X}$ , its TAIS and RIV.

to a  $(n, k, \lambda)$ -Cyclic Difference Set. Using tables of Cyclic Difference Sets we find all non-trivial magic  $(n, k, \lambda)$ -Ovals up to congruency for  $n \leq 40$ .

In Section 6 the rhombs of the regular  $2n$ -gon are partitioned into Ovals. Cyclic Difference Families are introduced and are shown to be equivalent to various Oval partitions; we also consider relevant integer partitions of the triangular number  $\binom{n}{2}$ .

In Section 7 we define  $u$ -equivalence for  $(n, k)$ -Ovals. The RIV's of two  $u$ -equivalent  $(n, k)$ -Ovals are closely related to each other. For each  $2 \leq n \leq 10$  we list all  $(n, k)$ -Ovals up to  $u$ -equivalence.

**2.  $(n, k)$ -Ovals, TAIS, the number of  $(n, k)$ -Ovals, multipliers, generating all  $(n, k)$ -Ovals**

Each fixed integer  $n \geq 2$  has associated with it  $\lfloor \frac{n}{2} \rfloor$  rhombs:  $\rho_1, \rho_2, \dots, \rho_{\lfloor \frac{n}{2} \rfloor}$ . For each  $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$  rhomb  $\rho_h$  is a parallelogram with all sides of unit length and with smaller face angle equal to  $h \times \frac{\pi}{n}$  radians;  $h$  is the *principal index* of the rhomb. The index of an adjacent face angle is  $n - h$ . The 7 rhombs for  $n = 15$  are shown in Fig. 2.

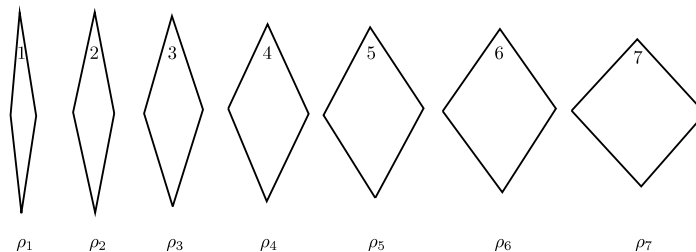


Fig. 2. The 7 rhombs, and their principal indices, corresponding to  $n = 15$ .

**Definitions 2.1.** Centro-symmetric, turning angle, Oval

- (1) A polygon is *centro-symmetric* if it is unchanged by a rotation of  $\pi$  radians (half a circle).
- (2) The *turning angle* at a vertex of a polygon is the supplement of the interior angle at that vertex.
- (3) An *Oval* is a centro-symmetric convex polygon all of whose sides are of unit length, and each of whose turning angles equals  $\ell \times \frac{\pi}{n}$  for some positive integer  $\ell$ .

Every Oval necessarily has an even number of sides, which are arranged in  $k$  parallel pairs.

**Definitions 2.2.**  $(n, k)$ -Oval, Turning Angle Index Sequence—TAIS

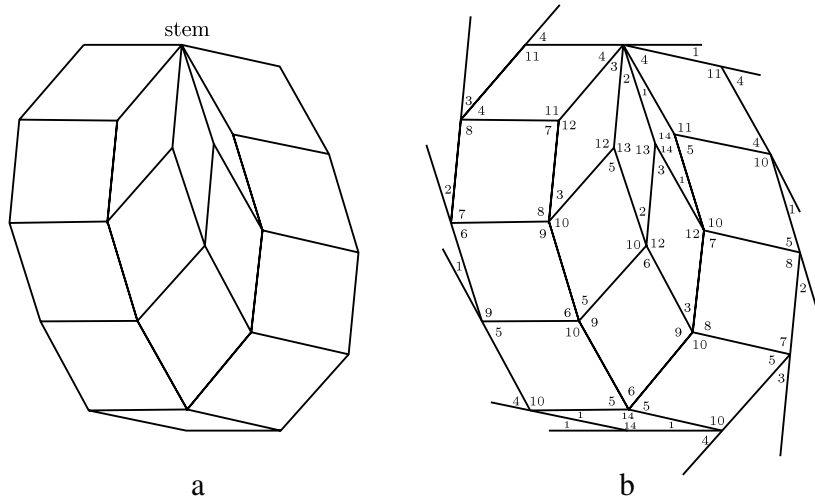
- (1) A  $(n, k)$ -Oval is an Oval with  $2k$  sides; it is described by the pair  $(n, k)$  and by its
- (2) *Turning Angle Index Sequence* (TAIS), a list of the turning angle indices for any  $k$  consecutive vertices.

We denote an arbitrary  $(n, k)$ -Oval by  $\mathcal{O}$  and specify a *stem* vertex of  $\mathcal{O}$ ; the TAIS of  $\mathcal{O}$  is then the list of turning angle indices at the  $k$  consecutive vertices taken in a counter-clockwise direction starting from the first vertex after the stem vertex.

**Remark 2.3.** The TAIS  $T$  of an  $(n, k)$ -Oval is simply a  $k$ -composition of  $n$ , i.e., an ordered list of  $k$  positive integers that sum to  $n$ :  $T = [t_1 t_2 \cdots t_k]$  with each  $t_i \geq 1$  and  $\sum_{i=1}^k t_i = n$ .

**Example 2.4.** The regular  $2n$ -gon,  $\{2n\}$ , is a  $(n, n)$ -Oval with TAIS= $\underbrace{[1 \ 1 \ \cdots \ 1]}_n$ . See Fig. 5 for a picture of the regular 12-gon,  $\{12\}$ .

**Example 2.5.**  $(n, k) = (15, 6)$ . In Fig. 3(a) we show the  $(15, 6)$ -Oval  $\mathcal{X}$  with TAIS  $T = [4 \ 3 \ 2 \ 1 \ 4 \ 1]$ . We write  $\mathcal{X} = \mathcal{O}(T) = \mathcal{O}([4 \ 3 \ 2 \ 1 \ 4 \ 1])$ . In (b) the turning angle index at each vertex of  $\mathcal{X}$  is shown, as well as all indices of the  $\binom{6}{2} = 15$  rhombs in  $\mathcal{X}$ . Note that the indices along the straight line at an ‘external’ vertex sum to  $n = 15$ , and the indices around an ‘internal’ vertex sum to  $2n = 30$ .



**Fig. 3.** See Fig. 1. The  $(15, 6)$ -Oval  $\mathcal{X}$  with TAIS  $T = [4 \ 3 \ 2 \ 1 \ 4 \ 1]$ .

Let  $S = \{s_1, s_2, \dots, s_k\}$  where  $0 \leq s_1 < s_2 < \dots < s_k$  be a  $k$ -subset of  $\mathbb{Z}_n$  with increasing elements. Throughout this paper the elements of  $S$  will always be written in increasing order.

Let  $U(n)$  denote the group of units modulo  $n$ , i.e., the multiplicative group of elements relatively prime to  $n$ .

**Definitions 2.6.**  $uS + z$ ,  $z$ -equivalent and  $\equiv_z$ , cyclically-equivalent and  $\equiv_{\text{cyc}}$

- (1)  $uS + z = \{us_1 + z, us_2 + z, \dots, us_k + z\} \subseteq \mathbb{Z}_n$  for  $u \in U(n)$  and  $z \in \mathbb{Z}_n$ .
- (2) Two  $k$ -subsets  $S$  and  $S'$  of  $\mathbb{Z}_n$  are  $z$ -equivalent,  $S \equiv_z S'$ , if there exists  $z \in \mathbb{Z}_n$  such that  $S = S' + z$ .
- (3) Two TAIS's  $T$  and  $T'$  are cyclically-equivalent,  $T \equiv_{\text{cyc}} T'$ , if  $T'$  is a cyclic permutation of  $T$ .

**Remark 2.7.** As an example of (3) above:

$$[t_1 \ t_2 \ t_3 \ t_4] \equiv_{\text{cyc}} [t_4 \ t_1 \ t_2 \ t_3] \equiv_{\text{cyc}} [t_3 \ t_4 \ t_1 \ t_2] \equiv_{\text{cyc}} [t_2 \ t_3 \ t_4 \ t_1].$$

Sometimes we use  $=$  in place of  $\equiv_z$  or  $\equiv_{\text{cyc}}$  for convenience.

Let  $\mathcal{S}^*(n, k)$  denote the set of all  $k$ -subsets  $S = \{s_1, s_2, \dots, s_k\} \subseteq \mathbb{Z}_n$  where  $0 \leq s_1 < s_2 < \dots < s_k$ . Then  $\equiv_z$  is an equivalence relation on  $\mathcal{S}^*(n, k)$ . We denote the set of equivalence classes of  $\equiv_z$  by  $\mathcal{S}_{\equiv_z}^*(n, k)$ . In an equivalence class  $[S]_{\equiv_z}$  or  $[S]$  we often use as representative the lowest member of  $[S]$  in lexicographic ordering.

Let  $\mathcal{T}^*(n, k)$  denote the set of all  $k$ -compositions of  $n$ , i.e., the set of TAIS  $T$  for all  $(n, k)$ -Ovals. Then  $\equiv_{\text{cyc}}$  is an equivalence relation on  $\mathcal{T}^*(n, k)$ . We denote the set of equivalence classes of  $\equiv_{\text{cyc}}$  by  $\mathcal{T}_{\equiv_{\text{cyc}}}^*(n, k)$ , and a typical equivalence class by  $[T]_{\equiv_{\text{cyc}}}$  or  $[T]$ .

**Theorem 2.12** gives a bijection between the sets  $\mathcal{S}_{\equiv_z}^*(n, k)$  and  $\mathcal{T}_{\equiv_{\text{cyc}}}^*(n, k)$ .

**Definitions 2.8.**  $\alpha(S)$  and  $\mathcal{O}(\alpha(S))$  or  $\mathcal{O}(T)$ ,  $\beta(T)$ .

Let  $S = \{s_1, s_2, \dots, s_k\} \subseteq \mathbb{Z}_n$  where  $0 \leq s_1 < s_2 < \dots < s_k$ .

(1)  $\alpha(S)$  is the ordered  $k$ -tuple

$$\alpha(S) = [s_2 - s_1, s_3 - s_2, \dots, s_k - s_{k-1}, s_1 - s_k],$$

(note that  $s_1 - s_k$  will be negative, it must be replaced with  $n - s_1 + s_k$ ). Then  $\mathcal{O}(\alpha(S)) = \mathcal{O}(T)$  is the  $(n, k)$ -Oval with TAIS  $\alpha(S) = T$ .

Let  $T = [t_1 t_2 \dots t_k]$  be the TAIS of a  $(n, k)$ -Oval.

(2)  $\beta(T)$  is the increasing  $k$ -subset of  $\mathbb{Z}_n$

$$\beta(T) = \beta([t_1 t_2 \dots t_k]) = \{0, t_1, t_1 + t_2, \dots, t_1 + t_2 + \dots + t_{k-1}\}.$$

**Remark 2.9.** See similar definitions on p. 221 of Beth et al. [3].

**Example 2.10.**  $(n, k) = (15, 6)$ . For the  $(15, 6)$ -Oval  $\mathcal{X}$  of Example 2.5 with TAIS  $T = [4 3 2 1 4 1]$  we have  $X = S = \beta(T) = \{0, 4, 7, 9, 10, 14\}$ , then  $\alpha(X) = T$ .

Compare the following theorem with Lemma 9.8, p. 221 of [3].

**Theorem 2.11.** Let  $S$  and  $S'$  be  $k$ -subsets of  $\mathbb{Z}_n$ . Then  $S \equiv_z S'$  if and only if  $\alpha(S) \equiv_{\text{cyc}} \alpha(S')$ .

**Proof.** Necessity: as usual, let  $S = \{s_1, s_2, \dots, s_k\}$  where  $0 \leq s_1 < s_2 < \dots < s_k$  and  $\alpha(S) = [s_2 - s_1, \dots, s_k - s_{k-1}, s_1 - s_k]$ . Suppose  $S \equiv_z S'$  then there exists  $z \in \mathbb{Z}_n$  with

$$\begin{aligned} S' &= S + z = \{s_1 + z, s_2 + z, \dots, s_k + z\} \\ &= \{s_i + z, s_{i+1} + z, \dots, s_k + z, s_1 + z, s_2 + z, \dots, s_{i-1} + z\} \end{aligned}$$

where  $0 \leq s_i + z < s_{i+1} + z < \dots < s_{i-1} + z$  is an increasing sequence for some  $i = 1, 2, \dots, k$ . So

$$\begin{aligned} \alpha(S') &= [s_{i+1} - s_i, \dots, s_1 - s_k, s_2 - s_1, \dots, s_{i-1} - s_{i-2}, s_i - s_{i-1}] \\ &\equiv_{\text{cyc}} [s_2 - s_1, \dots, s_{i-1} - s_{i-2}, s_i - s_{i-1}, s_{i+1} - s_i, \dots, s_1 - s_k] \\ &= \alpha(S), \text{ as required.} \end{aligned}$$

Sufficiency: if  $\alpha(S) \equiv_{\text{cyc}} \alpha(S')$  then  $\alpha(S')$  is a cyclic permutation of  $\alpha(S)$ . Without loss of generality let  $\alpha(S) = [t_1 t_2 \dots t_k]$  and  $\alpha(S') = [t_i t_{i+1} \dots t_k t_1 \dots t_{i-1}]$  for some  $i = 1, 2, \dots, k$ . Then  $\beta(\alpha(S)) = \{0, t_1, t_1 + t_2, \dots, t_1 + \dots + t_{k-1}\}$  and

$$\begin{aligned} \beta(\alpha(S')) &= \{0, t_i, t_i + t_{i+1}, \dots, t_i + \dots + t_k + t_1 + \dots + t_{i-2}\} \\ &= \beta(\alpha(S)) + (t_i + \dots + t_k) \\ &\equiv_z \beta(\alpha(S)). \end{aligned}$$

So  $\beta(\alpha(S')) \equiv_z \beta(\alpha(S))$ , but from Definition 2.8 we have  $\beta(\alpha(S)) = S - s_1 \equiv_z S$  for any  $S$ , and so  $S \equiv_z S'$  as required.  $\square$

**Theorem 2.12.** Let  $\alpha_{\equiv} : \mathcal{S}_{\equiv_z}^*(n, k) \leftrightarrow \mathcal{T}_{\equiv_{\text{cyc}}}^*(n, k)$  be given by  $\alpha_{\equiv}([S]) \leftrightarrow [\alpha(S)]$ . Then  $\alpha_{\equiv}$  is a bijection, and  $|\mathcal{S}_{\equiv_z}^*(n, k)| = |\mathcal{T}_{\equiv_{\text{cyc}}}^*(n, k)|$ .

**Remark 2.13.** Geometrically speaking, if two TAIS's  $T$  and  $T'$  are cyclically-equivalent, then the Ovals  $\mathcal{O}(T)$  and  $\mathcal{O}(T')$  can be 'moved' to one another in the plane using translations and rotations, a reflection is not required; we write  $\mathcal{O}(T) = \mathcal{O}(T')$ . The converse is also true. Thus  $T \equiv_{\text{cyc}} T'$  if and only if  $\mathcal{O}(T) = \mathcal{O}(T')$ .

**Definitions 2.14.**  $\mathcal{O}^*(n, k), \mathcal{O}(n, k)$

- (1)  $\mathcal{O}^*(n, k)$  is the set of  $(n, k)$ -Ovals equivalent up to translations and rotations.
- (2)  $\mathcal{O}(n, k) = |\mathcal{O}^*(n, k)|$  is the number of  $(n, k)$ -Ovals equivalent up to translations and rotations.

Each Oval in  $\mathcal{O}^*(n, k)$  has associated with it an equivalence class  $[T]$  in  $\mathcal{T}_{\equiv_{\text{cyc}}}^*(n, k)$ , and conversely each equivalence class  $[T]$  in  $\mathcal{T}_{\equiv_{\text{cyc}}}^*(n, k)$  gives an Oval  $\mathcal{O}(T)$  in  $\mathcal{O}^*(n, k)$ . So  $\mathcal{O}(n, k) = |\mathcal{T}_{\equiv_{\text{cyc}}}^*(n, k)|$ . This function is well-known to be the number of necklaces of size  $n$  with  $k$  white and  $n - k$  black beads; for an explicit calculation of  $\mathcal{O}(n, k)$  see p. 468 of Van Lint and Wilson [10]. Thus, letting  $\text{gcd}(n, k)$  denote the greatest common divisor of  $n$  and  $k$ , and  $\phi(x)$  denote Euler's totient function, we have the following.

**Theorem 2.15.** For  $n \geq 2$  and  $k \geq 2$ , the number of  $(n, k)$ -Ovals is

$$\mathcal{O}(n, k) = \frac{1}{n} \sum_{d | \text{gcd}(n, k)} \phi(d) \binom{\frac{n}{d}}{\frac{k}{d}}. \tag{1}$$

2.1. Multipliers, generating all  $(n, k)$ -Ovals

We wish to generate all Ovals in  $\mathcal{O}^*(n, k)$ . To do this we find a representative of each equivalence class  $[S]$  in  $\mathcal{S}_{\equiv_z}^*(n, k)$  and then use Theorem 2.12 to find a representative of each equivalence class  $[T]$  in  $\mathcal{T}_{\equiv_{\text{cyc}}}^*(n, k)$ .

**Definitions 2.16.** Multiplier  $m$  and  $\text{mult}(S)$ ,  $\text{mult}(\mathcal{O})$ .

Let  $S$  be a  $k$ -subset of  $\mathbb{Z}_n$ .

(1)  $m \in U(n)$  is a multiplier of  $S$  if  $S \equiv_z mS$ , i.e., if there exists  $z \in \mathbb{Z}_n$  with  $S = mS + z$ . The set of multipliers of  $S$  is  $\text{mult}(S)$ .

Let  $\mathcal{O}(T)$  be a  $(n, k)$ -Oval with TAIS  $T$ .

(2)  $m \in U(n)$  is a multiplier of  $\mathcal{O}(T)$  if  $m$  is a multiplier of  $S = \beta(T)$ . The set of multipliers of  $\mathcal{O}(T)$  is  $\text{mult}(\mathcal{O}(T)) = \text{mult}(S)$ .

**Remark 2.17.** See Chapter VI of [3] for examples of how multipliers are used in the theory of Cyclic Difference Sets; see also Section 5 of this paper. The set  $\text{mult}(S)$  is a subgroup of  $U(n)$ , and if  $S \equiv_z S'$  then  $\text{mult}(S) = \text{mult}(S')$ . Let  $T$  and  $T'$  be two different TAIS of an  $(n, k)$ -Oval  $\mathcal{O}$ . Then  $T \equiv_{\text{cyc}} T'$  and so  $\beta(T) \equiv_z \beta(T')$  by Theorem 2.11, and then  $\text{mult}(\beta(T)) = \text{mult}(\beta(T'))$ . Hence  $\text{mult}(\mathcal{O})$  is independent of the TAIS of  $\mathcal{O}$ .

**Example 2.18.**  $(n, k) = (15, 6)$ . For the  $(15, 6)$ -Oval  $\mathcal{X}$  of Examples 2.5 and 2.10 we have  $X = \{0, 4, 7, 9, 10, 14\}$  and so  $\text{mult}(\mathcal{X}) = \text{mult}(X) = \{1\}$ , the trivial group. For an example of a 6-set of  $\mathbb{Z}_{15}$  with non-trivial multiplier group consider  $Y = \{0, 1, 4, 7, 10, 13\}$ , here  $\text{mult}(Y) = \{1, 4, 7, 13\}$ .

Now  $m \in \text{mult}(S)$  if and only if  $S \equiv_z mS$ . Hence the number of  $z$ -inequivalent sets in  $\{uS : u \in U(n)\}$  equals the index of  $\text{mult}(S)$  in  $U(n)$ , i.e., equals  $|U(n) : \text{mult}(S)| = \frac{|U(n)|}{|\text{mult}(S)|}$ .

As an example of how to generate all Ovals in  $\mathcal{O}^*(n, k)$  we generate all Ovals in  $\mathcal{O}^*(7, 3)$ .

We have  $U(7) = \{1, 2, 3, 4, 5, 6\}$  and so  $|U(7)| = 6$ .

Start with  $A = \{0, 1, 2\}$ . So  $\text{mult}(A) = \{1, -1\}$  and  $|U(7) : \text{mult}(A)| = 3$ . The 3 cosets of  $\text{mult}(A)$  in  $U(7)$  are  $\text{mult}(A)$ ,  $2 \text{mult}(A)$ , and  $3 \text{mult}(A)$ . Hence the 3  $z$ -inequivalent sets in  $\{uA : u \in U(n)\}$  are  $A_1 = A$ ,  $A_2 = 2A = \{0, 2, 4\}$ , and  $A_3 = 3A = \{0, 3, 6\} \equiv_z \{0, 1, 4\}$ .

Then choose  $A' = \{0, 1, 3\}$  from  $\mathcal{S}^*(7, 3) \setminus ([A_1] \cup [A_2] \cup [A_3])$ . We have  $\text{mult}(A') = \{1, 2, 4\}$  and  $|U(7) : \text{mult}(A')| = 2$ . The 2 cosets of  $\text{mult}(A')$  in  $U(7)$  are  $\text{mult}(A')$  and  $3 \text{mult}(A')$ . Hence the 2  $z$ -inequivalent sets in  $\{uA' : u \in U(n)\}$  are  $A'_1 = A'$  and  $A'_2 = 3A' = \{3, 5, 6\} \equiv_z \{0, 1, 5\}$ .

Now  $\mathcal{S}^*(7, 3) \setminus ([A_1] \cup [A_2] \cup [A_3] \cup [A'_1] \cup [A'_2]) = \emptyset$ , so we stop. See Example 2.19.

**Example 2.19.**  $(n, k) = (7, 3)$ . Eq. (1) gives  $\mathcal{O}(7, 3) = |\mathcal{T}_{\equiv_{\text{cyc}}}^*(7, 3)| = \frac{1}{7}\phi(1) \binom{7}{3} = 5$ . Representatives of the 5 equivalence classes in both  $\mathcal{S}_{\equiv_z}^*(7, 3)$  and  $\mathcal{T}_{\equiv_{\text{cyc}}}^*(7, 3)$ , and the bijection between them, are given in the table below. The 5  $(7, 3)$ -Ovals up to translations and rotations are  $\mathcal{O}^*(7, 3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4), \mathcal{O}(T_5)\}$ , see Fig. 4. We will see that multiplier  $-1$  plays an important role in this paper. We use ' $A_i$ ' for a set with multiplier  $-1$ , and ' $B_i$ ' for a set without multiplier  $-1$ .

$S$	$T$	$\text{mult}(S)$	$\frac{ U(7) }{ \text{mult}(S) }$
$A_1 = \{0, 1, 2\}$	$\leftrightarrow T_1 = [1 \ 1 \ 5]$	$\{1, -1\}$	3
$A_2 = \{0, 2, 4\}$	$\leftrightarrow T_2 = [2 \ 2 \ 3]$	$\{1, -1\}$	
$A_3 = \{0, 1, 4\}$	$\leftrightarrow T_3 = [1 \ 3 \ 3]$	$\{1, -1\}$	
$B_1 = \{0, 1, 3\}$	$\leftrightarrow T_4 = [1 \ 2 \ 4]$	$\{1, 2, 4\}$	2
$B_2 = \{0, 1, 5\}$	$\leftrightarrow T_5 = [1 \ 4 \ 2]$	$\{1, 2, 4\}$	

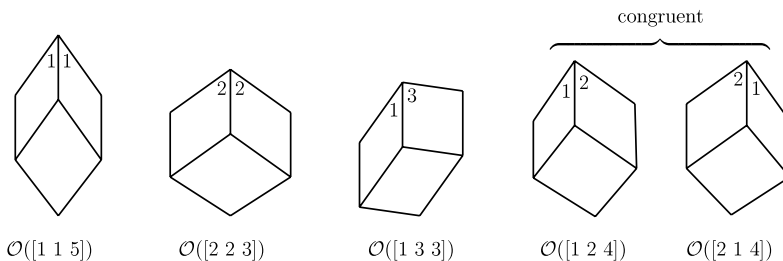


Fig. 4. The  $\mathcal{O}(7, 3) = 5$   $(7, 3)$ -Ovals up to translations and rotations. The last 2 form a congruent enantiomorphic pair.

It is clear how to generalize Example 2.19 to generate all Ovals in  $\mathcal{O}^*(n, k)$ , i.e., all  $(n, k)$ -Ovals up to translations and rotations, for an arbitrary  $(n, k)$  starting with  $A = \{0, 1, \dots, k - 1\}$ .

**3. Multiplier  $-1$ , reversible  $T$ , congruent Ovals, various counts**

In this section we consider multiplier  $-1$  of an  $(n, k)$ -Oval  $\mathcal{O}$ . We will return to consideration of multiplier  $-1$  in Section 5. Let  $T = [t_1 t_2 \dots t_k]$  be a TAIS of an  $(n, k)$ -Oval  $\mathcal{O}$ .

**Definitions 3.1.**  $\overleftarrow{T} = [t_k t_{k-1} \dots t_1]$  is the reverse of  $T$ .

**Lemma 3.2.** Let  $S$  and  $S'$  be  $k$ -subsets of  $\mathbb{Z}_n$ . Then

- (i)  $\alpha(-S) \equiv_{\text{cyc}} \overleftarrow{\alpha(S)}$ .
- (ii)  $S \equiv_z -S'$  if and only if  $\alpha(S) \equiv_{\text{cyc}} \overleftarrow{\alpha(S')}$ .

**Proof.** (i) Let  $S = \{s_1, s_2, \dots, s_k\}$ , where  $0 \leq s_1 < s_2 < \dots < s_k$ . Then  $-S = \{-s_1, -s_2, \dots, -s_k\} = \{n-s_1, n-s_2, \dots, n-s_k\} = \{n-s_k, n-s_{k-1}, \dots, n-s_2, n-s_1\}$ , in increasing order. So  $\alpha(-S) = [s_k - s_{k-1}, \dots, s_2 - s_1, s_1 - s_k] \equiv_{\text{cyc}} [s_1 - s_k, s_k - s_{k-1}, \dots, s_2 - s_1] = \overleftarrow{\alpha(S)}$ .

(ii) Necessity: let  $S \equiv_z -S'$  then  $\alpha(S) \equiv_{\text{cyc}} \alpha(-S') \equiv_{\text{cyc}} \overleftarrow{\alpha(S')}$  using Theorem 2.11 and then part (i) above.

Sufficiency: let  $\alpha(S) \equiv_{\text{cyc}} \overleftarrow{\alpha(S')}$  then  $\alpha(S) \equiv_{\text{cyc}} \alpha(-S')$  by part (i) applied to  $S'$ , and so  $S \equiv_z -S'$  by Theorem 2.11.  $\square$

**Definitions 3.3.** TAIS  $T$  is reversible if it is cyclically-equivalent to its reverse, i.e., if  $T \equiv_{\text{cyc}} \overleftarrow{T}$ , (equivalently,  $T \in [\overleftarrow{T}]$  or  $\overleftarrow{T} \in [T]$ ).

**Theorem 3.4.** Let  $S$  be a  $k$ -subset of  $\mathbb{Z}_n$ . Then  $-1 \in \text{mult}(S)$  if and only if  $\alpha(S)$  is reversible.

**Proof.** Now  $-1 \in \text{mult}(S)$  if and only if  $S \equiv_z -S$ , if and only if  $\alpha(S) \equiv_{\text{cyc}} \overleftarrow{\alpha(S)}$ , if and only if  $\alpha(S)$  is reversible.  $\square$

**Definitions 3.5.**  $\mathcal{O}(n, k; -1)$ ,  $\mathcal{O}(n, k; \overline{-1})$

- (1)  $\mathcal{O}(n, k; -1)$  is the number of  $(n, k)$ -Ovals with  $-1$  as a multiplier.
- (2)  $\mathcal{O}(n, k; \overline{-1})$  is the number of  $(n, k)$ -Ovals without  $-1$  as a multiplier.

A  $k$ -reverse of  $n$  is a reversible  $k$ -composition of  $n$ . In McSorley [6] using Polya Theory we count the number of  $k$ -reverses of  $n$  up to cyclic permutation; this number is denoted by  $\mathcal{R}_{\equiv}(n, k)$ . From Theorem 3.4 we have  $\mathcal{O}(n, k; -1) = \mathcal{R}_{\equiv}(n, k)$ .

**Theorem 3.6.** For  $n \geq 2$  and  $k \geq 2$ , the number of  $(n, k)$ -Ovals with  $-1$  as a multiplier is

$$\mathcal{O}(n, k; -1) = \begin{cases} \binom{\frac{n-2}{2}}{\frac{k-1}{2}}, & \text{if } n \text{ is even and } k \text{ is odd;} \\ \binom{\frac{n-1}{2}}{\frac{k-1}{2}}, & \text{if } n \text{ is odd and } k \text{ is odd;} \\ \binom{\frac{n}{2}}{\frac{k}{2}}, & \text{if } n \text{ is even and } k \text{ is even;} \\ \binom{\frac{n-1}{2}}{\frac{k}{2}}, & \text{if } n \text{ is odd and } k \text{ is even.} \end{cases}$$

For a given TAIS  $T$  we obtain Oval  $\mathcal{O}(\overleftarrow{T})$  from Oval  $\mathcal{O}(T)$  by reflecting  $\mathcal{O}(T)$  in a straight line that (for simplicity) does not intersect  $\mathcal{O}(T)$ . We denote the reflection of  $\mathcal{O}$  by  $\overleftarrow{\mathcal{O}}$ .

When Ovals  $\mathcal{O}(T)$  and  $\mathcal{O}(\overleftarrow{T})$  cannot be moved to one another using only translations and rotations, we say they are enantiomorphs of each other. In this case  $\mathcal{O}(T) \neq \mathcal{O}(\overleftarrow{T})$  and a reflection is required to move  $\mathcal{O}(T)$  to  $\mathcal{O}(\overleftarrow{T})$  and vice-versa. (Oval  $\mathcal{O}(T)$  is congruent to  $\mathcal{O}(\overleftarrow{T})$ ; see Section 3.1.) These comments and Theorem 3.4 give the following.

**Theorem 3.7.** Let  $\mathcal{O}(T)$  be a  $(n, k)$ -Oval.

- (i)  $\mathcal{O}(T)$  has multiplier  $-1$  if and only if  $T$  is reversible, if and only if  $\mathcal{O}(T) = \mathcal{O}(\overleftarrow{T})$ .
- (ii)  $\mathcal{O}(T)$  does not have multiplier  $-1$  if and only if  $T$  is not reversible, if and only if  $\mathcal{O}(T) \neq \mathcal{O}(\overleftarrow{T})$ . Such Ovals occur in  $\{\mathcal{O}(T), \mathcal{O}(\overleftarrow{T})\}$  (congruent) enantiomorphic pairs in  $\mathcal{O}^*(n, k)$ . (Hence there is an even number of Ovals in  $\mathcal{O}^*(n, k)$  without multiplier  $-1$ .)

**Example 3.8.**  $(n, k) = (7, 3)$ . See Example 2.19.

$\mathcal{O}^*(7, 3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4), \mathcal{O}(T_5)\}$ , and Theorem 3.6 gives  $\mathcal{O}(7, 3; -1) = \binom{3}{1} = 3$ .

If  $i = 1, 2, \text{ or } 3$ , then  $-1 \in \text{mult}(\mathcal{O}(T_i))$  and so  $T_i \equiv_{\text{cyc}} \overleftarrow{T_i}$ ; eg., for  $i = 1$  we have  $[1\ 1\ 5] \equiv_{\text{cyc}} [5\ 1\ 1] (= [\overleftarrow{1\ 1\ 5}])$ .

If  $i = 4, \text{ or } 5$ , then  $-1 \notin \text{mult}(\mathcal{O}(T_i))$  and so  $T_i \not\equiv_{\text{cyc}} \overleftarrow{T_i}$ ; eg., for  $i = 4$  we have  $[1\ 2\ 4] \not\equiv_{\text{cyc}} [4\ 2\ 1] (= [\overleftarrow{1\ 2\ 4}])$ .

The pair  $\{\mathcal{O}(T_4), \mathcal{O}(T_5)\} = \{\mathcal{O}(T_4), \mathcal{O}(\overleftarrow{T_4})\}$  is a (congruent) enantiomorphic pair referred to in Theorem 3.7(ii).

3.1. Congruent Ovals

**Definitions 3.9.** Congruent and  $\equiv_c$

- (1) Two  $k$ -subsets  $S$  and  $S'$  of  $\mathbb{Z}_n$  are congruent,  $S \equiv_c S'$ , if  $S \equiv_z S'$  or  $S \equiv_z -S'$ .
- (2) Two TAIS  $T$  and  $T'$  are congruent,  $T \equiv_c T'$ , if  $T \equiv_{\text{cyc}} T'$  or  $T \equiv_{\text{cyc}} \overleftarrow{T'}$ .
- (3) Two  $(n, k)$ -Ovals  $\mathcal{O}$  and  $\mathcal{O}'$  are congruent,  $\mathcal{O} \equiv_c \mathcal{O}'$ , if  $\mathcal{O} = \mathcal{O}'$  or  $\mathcal{O} = \mathcal{O}'$ , i.e., if  $\mathcal{O}$  can be moved to  $\mathcal{O}'$  by a sequence of translations, rotations, or reflections, (isometries).

Then, from Theorem 2.11 and Lemma 3.2, we have the following.

**Theorem 3.10.** Let  $S$  and  $S'$  be  $k$ -subsets of  $\mathbb{Z}_n$ . Then  $S \equiv_c S'$  if and only if  $\alpha(S) \equiv_c \alpha(S')$ , if and only if  $\mathcal{O}(\alpha(S)) \equiv_c \mathcal{O}(\alpha(S'))$ .

**Definitions 3.11.**  $\text{Mult}(S) = \text{mult}(S) \cup -\text{mult}(S)$ .

**Remark 3.12.** It is straightforward to show that  $\text{Mult}(S)$  is a subgroup of  $U(n)$ . If  $-1 \in \text{mult}(S)$  then  $\text{Mult}(S) = \text{mult}(S)$ , and if  $-1 \notin \text{mult}(S)$  then  $|\text{Mult}(S)| = 2 |\text{mult}(S)|$ .

**Definitions 3.13.**  $\mathcal{O}_c^*(n, k), \mathcal{O}_c(n, k)$

- (1)  $\mathcal{O}_c^*(n, k)$  is the set of  $(n, k)$ -Ovals up to congruency.
- (2)  $\mathcal{O}_c(n, k) = |\mathcal{O}_c^*(n, k)|$  is the number of  $(n, k)$ -Ovals up to congruency.

In order to generate the set  $\mathcal{O}_c^*(n, k)$  for an arbitrary  $(n, k)$  we may use the procedure in Section 2.1 to find  $\mathcal{O}^*(n, k)$  and then combine congruent enantiomorphic pairs of Ovals; see Theorem 3.7(ii). Alternatively, we may use this procedure with the group  $\text{mult}(S)$  replaced by  $\text{Mult}(S)$ .

**Example 3.14.**  $(n, k) = (7, 3)$ . See Examples 2.19 and 3.8.

To find  $\mathcal{O}_c^*(7, 3)$  using the first method mentioned above we start with  $\mathcal{O}^*(7, 3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4), \mathcal{O}(\overleftarrow{T_4})\}$  and combine the last 2 Ovals into a single congruency class to give  $\mathcal{O}_c^*(7, 3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4)\}$ .

Using the second method, the procedure of Section 2.1 with  $\text{mult}(S)$  replaced by  $\text{Mult}(S)$  gives the following table:

$S$	$T$	$\text{Mult}(S)$	$\frac{ U(7) }{ \text{Mult}(S) }$
$A_1 = \{0, 1, 2\}$	$\leftrightarrow T_1 = [1\ 1\ 5]$	$\{1, -1\}$	3
$A_2 = \{0, 2, 4\}$	$\leftrightarrow T_2 = [2\ 2\ 3]$	$\{1, -1\}$	
$A_3 = \{0, 1, 4\}$	$\leftrightarrow T_3 = [1\ 3\ 3]$	$\{1, -1\}$	
$B_1 = \{0, 1, 3\}$	$\leftrightarrow T_4 = [1\ 2\ 4]$	$U(7)$	1

This also gives  $\mathcal{O}_c^*(7, 3) = \{\mathcal{O}(T_1), \mathcal{O}(T_2), \mathcal{O}(T_3), \mathcal{O}(T_4)\}$ , the set of all  $(7, 3)$ -Ovals up to congruency.

3.2.  $\mathcal{O}_c(n, k)$ ,  $\mathcal{O}_c(n, k; -1)$ , and  $\mathcal{O}_c(n, k; \overline{-1})$ **Definitions 3.15.**  $\mathcal{O}_c(n, k; -1)$ ,  $\mathcal{O}_c(n, k; \overline{-1})$ 

- (1)  $\mathcal{O}_c(n, k; -1)$  is the number of  $(n, k)$ -Ovals with  $-1$  as a multiplier, up to congruency.  
 (2)  $\mathcal{O}_c(n, k; \overline{-1})$  is the number of  $(n, k)$ -Ovals without  $-1$  as a multiplier, up to congruency.

**Lemma 3.16.**

$$\mathcal{O}_c(n, k) = \frac{1}{2} \left( \mathcal{O}(n, k) + \mathcal{O}(n, k; -1) \right).$$

**Proof.**

$$\begin{aligned} \mathcal{O}_c(n, k) &= \mathcal{O}_c(n, k; -1) + \mathcal{O}_c(n, k; \overline{-1}) \\ &= \mathcal{O}(n, k; -1) + \frac{1}{2} \mathcal{O}(n, k; \overline{-1}) \\ &= \mathcal{O}(n, k; -1) + \frac{1}{2} (\mathcal{O}(n, k) - \mathcal{O}(n, k; -1)) \\ &= \frac{1}{2} (\mathcal{O}(n, k) + \mathcal{O}(n, k; -1)). \end{aligned}$$

At the second line we use  $\mathcal{O}(n, k; -1) = \mathcal{O}_c(n, k; -1)$  because if  $\mathcal{O}$  and  $\mathcal{O}'$  both have  $-1$  as a multiplier then, from Definition 3.9(3) and Theorem 3.7(i), we have  $\mathcal{O} = \mathcal{O}'$  if and only if  $\mathcal{O} \equiv_c \mathcal{O}'$ . And  $\mathcal{O}_c(n, k; \overline{-1}) = \frac{1}{2} \mathcal{O}(n, k; \overline{-1})$  comes directly from Theorem 3.7(ii).  $\square$

Recall that  $\mathcal{O}(n, k)$  is given explicitly in Eq. (1).

**Theorem 3.17.** For  $n \geq 2$  and  $k \geq 2$ , the number of  $(n, k)$ -Ovals up to congruency is

$$\mathcal{O}_c(n, k) = \begin{cases} \frac{1}{2} \left( \mathcal{O}(n, k) + \binom{\frac{n-2}{2}}{\frac{k-1}{2}} \right), & \text{if } n \text{ is even and } k \text{ is odd;} \\ \frac{1}{2} \left( \mathcal{O}(n, k) + \binom{\frac{n-1}{2}}{\frac{k-1}{2}} \right), & \text{if } n \text{ is odd and } k \text{ is odd;} \\ \frac{1}{2} \left( \mathcal{O}(n, k) + \binom{\frac{n}{2}}{\frac{k}{2}} \right), & \text{if } n \text{ is even and } k \text{ is even;} \\ \frac{1}{2} \left( \mathcal{O}(n, k) + \binom{\frac{n-1}{2}}{\frac{k}{2}} \right), & \text{if } n \text{ is odd and } k \text{ is even.} \end{cases}$$

Theorem 3.6 now gives the following.

**Theorem 3.18.** For  $n \geq 2$  and  $k \geq 2$ , the number of  $(n, k)$ -Ovals without  $-1$  as a multiplier up to congruency is

$$\mathcal{O}_c(n, k; \overline{-1}) = \begin{cases} \frac{1}{2} \left( \mathcal{O}(n, k) - \binom{\frac{n-2}{2}}{\frac{k-1}{2}} \right), & \text{if } n \text{ is even and } k \text{ is odd;} \\ \frac{1}{2} \left( \mathcal{O}(n, k) - \binom{\frac{n-1}{2}}{\frac{k-1}{2}} \right), & \text{if } n \text{ is odd and } k \text{ is odd;} \\ \frac{1}{2} \left( \mathcal{O}(n, k) - \binom{\frac{n}{2}}{\frac{k}{2}} \right), & \text{if } n \text{ is even and } k \text{ is even;} \\ \frac{1}{2} \left( \mathcal{O}(n, k) - \binom{\frac{n-1}{2}}{\frac{k}{2}} \right), & \text{if } n \text{ is odd and } k \text{ is even.} \end{cases}$$



**Table 1**

Values of  $\mathcal{O}_c(n, k)$ ,  $\mathcal{O}_c(n, k; -1)$ , and  $\mathcal{O}_c(n, k; \overline{-1})$  for  $2 \leq k \leq n \leq 10$ , and of  $\mathcal{O}_c(n)$ ,  $\mathcal{O}_c(n; -1)$ , and  $\mathcal{O}_c(n; \overline{-1})$  for  $2 \leq n \leq 10$ .

(a) $\mathcal{O}_c(n, k)$										
$n/k$	2	3	4	5	6	7	8	9	10	$\mathcal{O}_c(n)$
2	1									1
3	1	1								2
4	2	1	1							4
5	2	2	1	1						6
6	3	3	3	1	1					11
7	3	4	4	3	1	1				16
8	4	5	8	5	4	1	1			28
9	4	7	10	10	7	4	1	1		44
10	5	8	16	16	16	8	5	1	1	76
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
(b) $\mathcal{O}_c(n, k; -1)$										
$n/k$	2	3	4	5	6	7	8	9	10	$\mathcal{O}_c(n; -1)$
2	1									1
3	1	1								2
4	2	1	1							4
5	2	2	1	1						6
6	3	2	3	1	1					10
7	3	3	3	3	1	1				14
8	4	3	6	3	4	1	1			22
9	4	4	6	6	4	4	1	1		30
10	5	4	10	6	10	4	5	1	1	46
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
(c) $\mathcal{O}_c(n, k; \overline{-1})$										
$n/k$	2	3	4	5	6	7	8	9	10	$\mathcal{O}_c(n; \overline{-1})$
2	0									0
3	0	0								0
4	0	0	0							0
5	0	0	0	0						0
6	0	1	0	0	0					1
7	0	1	1	0	0	0				2
8	0	2	2	2	0	0	0			6
9	0	3	4	4	3	0	0	0		14
10	0	4	6	10	6	4	0	0	0	30
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

See Table 1(a). The triangle of values of  $\mathcal{O}_c(n, k)$  when read row-by-row gives sequence A052307 in the Online Encyclopedia of Integer Sequences [7].

See Table 1(b). The triangle of values of  $\mathcal{O}_c(n, k; -1) = \mathcal{O}(n, k; -1)$  (see Theorem 3.6) is equal to the triangle of sequence A119963 in [7] (with the first two columns of 1's removed). So  $\mathcal{O}_c(n, k; -1)$  gives the first combinatorial interpretation of sequence A119963 in [7]. Thus (ignoring the first two columns of 1's) the  $(n, k)$  term in the triangle of sequence A119963 is the number of  $(n, k)$ -Ovals with  $-1$  as a multiplier, up to congruency. For the sequence of row sums of the triangle of sequence A119963 see sequence A029744, and the comment 'Necklaces with  $n$  beads that are the same when turned over'.

See Table 1(c). When the triangle of values of  $\mathcal{O}_c(n, k; \overline{-1})$  is read row-by-row we obtain a new sequence, see sequence A180472 in [7]. For the sequence of row sums of this triangle see sequence A059076: 'Number of orientable necklaces with  $n$  beads and two colors; i.e., turning over the necklace does not leave it unchanged'.

**Example 3.19.**  $(n, k) = (7, 3)$ . From Example 3.14 the number of  $(7, 3)$ -Ovals up to congruency is 4. Theorem 3.17 gives  $\mathcal{O}_c(7, 3) = \frac{1}{2}(\mathcal{O}(7, 3) + \binom{3}{1}) = \frac{1}{2}(5 + 3) = 4$ , also. Of these 4 Ovals, 3 have  $-1$  as a multiplier, and 1 does not. Theorem 3.6 gives  $\mathcal{O}_c(7, 3; -1) = \binom{3}{1} = 3$ , and Theorem 3.18 gives  $\mathcal{O}_c(7, 3; \overline{-1}) = \frac{1}{2}(\mathcal{O}(7, 3) - \binom{3}{1}) = \frac{1}{2}(5 - 3) = 1$ . Thus all counts for  $(n, k) = (7, 3)$  from Example 3.14 are confirmed.

**4. Rhombic inventory vector, all  $(n, k)$ -Ovals for  $n \leq 10$**

We use  $\subseteq_m$  to denote containment in multisets. For example, if multiset  $M = \{1, 1, 1, 2, 3, 3, 4, 4, 4, 4\}$  then  $L = \{1, 1, 1, 2, 4, 4\} \subseteq_m M$  but  $L' = \{1, 1, 1, 2, 2\} \not\subseteq_m M$ . We say that  $L$  is a multisubset of  $M$ . Further, we replace  $\underbrace{a, a, \dots, a}_b$  by  $a^b$ , so  $M = \{1^3, 2^1, 3^2, 4^4\}$ .

On p. 141 of Ball and Coxeter [1] it is proved that every  $(n, k)$ -Oval  $\mathcal{O}$ , with  $2 \leq k \leq n$ , can be tiled by a multiset of  $\binom{k}{2}$  rhombs chosen from  $\rho_1, \rho_2, \dots, \rho_{\lfloor \frac{n}{2} \rfloor}$ .

The regular  $2n$ -gon,  $\{2n\}$ , is a  $(n, n)$ -Oval with  $\text{TAIS} = \underbrace{[1 \ 1 \ \dots \ 1]}_n$ .

**Definitions 4.1.** The *Standard Rhombic Inventory*,  $\text{SRI}_{2n}$ , is the multiset of  $\binom{n}{2}$  rhombs that tile  $\{2n\}$ .

There are  $\lfloor \frac{n}{2} \rfloor$  different shapes of rhombs in  $\text{SRI}_{2n}$ ; see Section 2. When  $n$  is odd,  $\text{SRI}_{2n}$  contains  $n$  copies of each of the  $\frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$  shapes of rhomb,  $\rho_1, \rho_2, \dots, \rho_{\frac{n-1}{2}}$ . When  $n$  is even,  $\text{SRI}_{2n}$  contains  $n$  copies of each of the  $\frac{n}{2} - 1$  non-square rhombs,  $\rho_1, \rho_2, \dots, \rho_{\frac{n}{2}-1}$ , but only  $\frac{n}{2}$  copies of the square  $\rho_{\frac{n}{2}}$ .

For a fixed  $(n, k)$ -Oval  $\mathcal{O}$  let  $\lambda_h$  equal the number of rhombs in  $\mathcal{O}$  with principal index  $h$ .

**Definitions 4.2.** The *Rhombic Inventory Vector* (RIV) of Oval  $\mathcal{O}$ ,  $\text{RIV}(\mathcal{O})$ , is the vector  $(\lambda_1, \lambda_2, \dots, \lambda_{\lfloor \frac{n}{2} \rfloor})$  of length  $\lfloor \frac{n}{2} \rfloor$ .

The sum of the components in  $\text{RIV}(\mathcal{O})$  equals  $\binom{k}{2}$ .

**Example 4.3.**  $(n, k) = (15, 6)$ . See Figs. 1 and 3. The  $(15, 6)$ -Oval  $\mathcal{X}$  is tiled by  $\binom{6}{2} = 15$  rhombs. The rhomb  $\rho_4$  occurs twice in  $\mathcal{X}$ , so  $\lambda_4 = 2$ . We have  $\text{RIV}(\mathcal{X}) = (2, 1, 2, 2, 4, 2, 2)$ .

The RIV of a  $(n, k)$ -Oval can be derived from its TAIS by constructing its *Oval Index Triangle*, (OIT). The construction of an OIT is described below for our  $(15, 6)$ -Oval  $\mathcal{X}$ .

First we define the function  $r : \mathbb{Z}_n \setminus \{0\} \mapsto \mathbb{Z}_n \setminus \{0\}$ :

$$r(a) = \begin{cases} a & \text{if } a \leq \lfloor \frac{n}{2} \rfloor, \\ -a \text{ or } n - a & \text{if } a > \lfloor \frac{n}{2} \rfloor. \end{cases} \tag{2}$$

We extend the definition of  $r$  to multisets  $M$  as follows:  $r(M) = \{r(a) \mid a \in M\}$ .

The TAIS for  $\mathcal{X}$  is  $[4 \ 3 \ 2 \ 1 \ 4 \ 1]$ . To compute  $\text{RIV}(\mathcal{X})$ :

- (i) Delete the last turning angle index from the TAIS, thereby obtaining the sequence of indices for the upper interior face angles of the rhombs in the *receptacle* – the cluster of  $k - 1$  rhombs that are incident on the stem vertex of the Oval. ('Receptacle' is the term used by botanists to denote the part of a plant that holds the fruit.) We call this sequence the 'truncated TAIS'. The truncated TAIS for  $\mathcal{X}$  is  $[4 \ 3 \ 2 \ 1 \ 4]$ .
- (ii) The first row of the OIT equals the truncated TAIS. Below each pair of consecutive indices in the first row enter their sum in the second row:

$$\begin{array}{cccccc} 4 & 3 & 2 & 1 & 4 & \\ & 7 & 5 & 3 & 5 & \end{array}$$

- (iii) Let  $h_{i,j}$  denote the index in row  $i$  and position  $j$  of the triangle, where  $i \geq 3$ , and  $j = 1, 2, \dots, k - i$ , counting from the left. Now the indices at each interior vertex of a  $(n, k)$ -Oval sum to  $2n$ , so simple trigonometry gives:

$$h_{i+1,j} = h_{i,j} + h_{i,j+1} - h_{i-1,j+1}.$$

See the left-hand triangle in (iv) below.

- (iv) Apply function  $r$  to the indices of the left-hand triangle, i.e., replace index  $h > \lfloor \frac{n}{2} \rfloor$  by  $n - h$ . The OIT is now complete.

$$\begin{array}{cccccc} 4 & 3 & 2 & 1 & 4 & \\ & 7 & 5 & 3 & 5 & \\ & & 9 & 6 & 7 & \\ & & & 10 & 10 & \\ & & & & 14 & \end{array} \xrightarrow{r} \begin{array}{cccccc} 4 & 3 & 2 & 1 & 4 & \\ & 7 & 5 & 3 & 5 & \\ & & 6 & 6 & 7 & \\ & & & 5 & 5 & \\ & & & & 1 & \\ & & & & & \text{OIT} \end{array}$$

- (v) Now count the frequency of each principal index in the OIT to obtain  $\text{RIV}(\mathcal{X}) = (2, 1, 2, 2, 4, 2, 2)$ , as above.

Recall the definition of  $\alpha(S)$  from Definition 2.8(1).

**Definitions 4.4.**  $\delta(S)$ ,  $\text{OIT}(\alpha(S))$  or  $\text{OIT}(T)$ .

Let  $S = \{s_1, s_2, \dots, s_k\} \subseteq \mathbb{Z}_n$ .

(1)  $\delta(S) = \{s_j - s_i : 1 \leq i < j \leq k\}$  is a multiset of non-zero differences of  $S$ .

Note that  $|\delta(S)| = \binom{k}{2}$ .

(2)  $\text{OIT}(\alpha(S)) = \text{OIT}(T)$  is the multiset of indices in the OIT with first row  $[s_2 - s_1, s_3 - s_2, \dots, s_k - s_{k-1}]$ , the truncation of  $\alpha(S) = T$ .

**Lemma 4.5.** Let  $S = \{s_1, s_2, \dots, s_k\} \subseteq \mathbb{Z}_n$ . Then  $\text{OIT}(\alpha(S)) = r(\delta(S))$ .

**Proof.** Consider the triangle formed previously with  $h_{i,j}$  as the index in row  $i$  and position  $j$ , counting from the left, and let  $H$  denote the multiset of all such  $h_{i,j}$ .

We show for  $i = 1, 2, \dots, k - 1$ , and  $j = 1, 2, \dots, k - i$  that  $h_{i,j} = s_{i+j} - s_j \in \delta(S)$ , i.e., that the indices in row  $i$  of this triangle are the difference of two  $s$ 's  $\in S$  whose subscripts differ by  $i$ .

By definition of the triangle this is clearly true for  $i = 1, 2$ . Assume that the hypothesis is true for rows  $1, 2, \dots, i$ . Then, for  $i \geq 3$ :

$$\begin{aligned} h_{i+1,j} &= h_{i,j} + h_{i,j+1} - h_{i-1,j+1} \\ &= (s_{i+j} - s_j) + (s_{i+(j+1)} - s_{j+1}) - (s_{(i-1)+(j+1)} - s_{j+1}) \\ &= s_{(i+1)+j} - s_j \in \delta(S), \end{aligned}$$

using strong induction at the second line. Hence the induction goes through, and  $H \subseteq_m \delta(S)$ , but  $|H| = \binom{k}{2} = |\delta(S)|$ , and so  $H = \delta(S)$ . Now apply  $r$  to both sides of this equation to give the result.  $\square$

**Example 4.6.**  $(n, k) = (15, 6)$ . Our  $(15, 6)$ -Oval  $\mathcal{X}$  has TAIS  $T = [4 \ 3 \ 2 \ 1 \ 4 \ 1]$ . So  $X = \beta(T) = \{0, 4, 7, 9, 10, 14\}$ , giving  $\delta(X) = \{1^1, 2^1, 3^2, 4^2, 5^2, 6^1, 7^2, 9^1, 10^2, 14^1\}$ , and  $r(\delta(X)) = \{1^2, 2^1, 3^2, 4^2, 5^4, 6^2, 7^2\}$ . So  $\text{RIV}(\mathcal{X}) = (2, 1, 2, 2, 4, 2, 2)$ , as above.

**Remark 4.7.** It is straightforward to show that the multiset  $\text{OIT}(T)$  does not depend on how we truncated  $T$  to form the first row of the OIT.

#### 4.1. All $(n, k)$ -Ovals and their RIV's for $n \leq 10$

In Tables 2 and 3 we list and number all  $(n, k)$ -Ovals up to congruence, and their RIV's for  $2 \leq n \leq 10$ . We will refer to these Ovals by their numbers in later Sections.

### 5. Magic Ovals, cyclic difference sets, multiplier $-1$ , all magic $(n, k, \lambda)$ -Ovals for $n \leq 40$

Recall  $S = \{s_1, s_2, \dots, s_k\} \subseteq \mathbb{Z}_n$ , and  $r : \mathbb{Z}_n \setminus \{0\} \mapsto \mathbb{Z}_n \setminus \{0\}$  from Eq. (2), and  $\delta(S)$  from Definition 4.4(1); let  $M$  be a multiset with elements from  $\mathbb{Z}_n \setminus \{0\}$ . We need two more definitions.

**Definitions 5.1.**  $f_M(a)$ ,  $\Delta(S)$

(1)  $f_M(a)$  is the frequency of  $a \in M$ .

(2)  $\Delta(S) = \delta(S) \cup -\delta(S)$  is the multiset of non-zero differences of  $S$ .

Note that  $-\delta(S) = \{s_i - s_j : 1 \leq i < j \leq k\}$ , and  $|\delta(S)| = |\delta(S)| = \binom{k}{2}$ , and  $|\Delta(S)| = k(k - 1)$ .

**Lemma 5.2.** Let  $M$  be a multiset with elements from  $\mathbb{Z}_n \setminus \{0\}$ . Then  $r(M) = r(-M)$ .

**Proof.** Let  $n$  be even. Consider an occurrence of  $a \in M$ .

Suppose  $a \leq \lfloor \frac{n}{2} \rfloor$ . First, if  $a = \frac{n}{2}$  then  $r(a) = \frac{n}{2}$ . Now  $-a = \frac{n}{2} \in -M$  and  $r(-a) = \frac{n}{2}$  also. Thus element  $\frac{n}{2} \in M$  'contributes' the same element  $\frac{n}{2}$  to both multisets  $r(M)$  and  $r(-M)$ . Second, if  $a < \lfloor \frac{n}{2} \rfloor$  then  $r(a) = a$ . Now  $-a \in -M$  satisfies  $-a > \lfloor \frac{n}{2} \rfloor$  so  $r(-a) = -(-a) = a$ . So, again, element  $a \in M$  contributes the same element  $a$  to both  $r(M)$  and  $r(-M)$ .

Suppose  $a > \lfloor \frac{n}{2} \rfloor$ . Then  $r(a) = -a$ . Now  $-a \in -M$  satisfies  $-a < \lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor$  so  $r(-a) = -a$ . Thus, element  $a \in M$  contributes the same element  $-a$  to both  $r(M)$  and  $r(-M)$ .

**Table 2**

All  $(n, k)$ -Ovals up to congruence and their RIV's for  $2 \leq n \leq 7$ .

$\mathcal{O}_i$	$k$	TAIS	RIV
$n = 2$			
$\mathcal{O}_1$	2	[1 1]	(1)
$n = 3$			
$\mathcal{O}_1$	2	[1 2]	(1)
$\mathcal{O}_2$	3	[1 1 1]	(3)
$n = 4$			
$\mathcal{O}_1$	2	[1 3]	(1, 0)
$\mathcal{O}_2$	2	[2 2]	(0, 1)
$\mathcal{O}_3$	3	[1 1 2]	(2, 1)
$\mathcal{O}_4$	4	[1 1 1 1]	(4, 2)
$n = 5$			
$\mathcal{O}_1$	2	[1 4]	(1, 0)
$\mathcal{O}_2$	2	[2 3]	(0, 1)
$\mathcal{O}_3$	3	[1 1 3]	(2, 1)
$\mathcal{O}_4$	3	[1 2 2]	(1, 2)
$\mathcal{O}_5$	4	[1 1 1 2]	(3, 3)
$\mathcal{O}_6$	5	[1 1 1 1 1]	(5, 5)
$n = 6$			
$\mathcal{O}_1$	2	[1 5]	(1, 0, 0)
$\mathcal{O}_2$	2	[2 4]	(0, 1, 0)
$\mathcal{O}_3$	2	[3 3]	(0, 0, 1)
$\mathcal{O}_4$	3	[1 1 4]	(2, 1, 0)
$\mathcal{O}_5$	3	[1 2 3]	(1, 1, 1)
$\mathcal{O}_6$	3	[2 2 2]	(0, 3, 0)
$\mathcal{O}_7$	4	[1 1 1 3]	(3, 2, 1)
$\mathcal{O}_8$	4	[1 1 2 2]	(2, 3, 1)
$\mathcal{O}_9$	4	[1 2 1 2]	(2, 2, 2)
$\mathcal{O}_{10}$	5	[1 1 1 1 2]	(4, 4, 2)
$\mathcal{O}_{11}$	6	[1 1 1 1 1 1]	(6, 6, 3)
$n = 7$			
$\mathcal{O}_1$	2	[1 6]	(1, 0, 0)
$\mathcal{O}_2$	2	[2 5]	(0, 1, 0)
$\mathcal{O}_3$	2	[3 4]	(0, 0, 1)
$\mathcal{O}_4$	3	[1 1 5]	(2, 1, 0)
$\mathcal{O}_5$	3	[1 2 4]	(1, 1, 1)
$\mathcal{O}_6$	3	[1 3 3]	(1, 0, 2)
$\mathcal{O}_7$	3	[2 2 3]	(0, 2, 1)
$\mathcal{O}_8$	4	[1 1 1 4]	(3, 2, 1)
$\mathcal{O}_9$	4	[1 1 2 3]	(2, 2, 2)
$\mathcal{O}_{10}$	4	[1 2 1 3]	(2, 1, 3)
$\mathcal{O}_{11}$	4	[1 2 2 2]	(1, 3, 2)
$\mathcal{O}_{12}$	5	[1 1 1 1 3]	(4, 3, 3)
$\mathcal{O}_{13}$	5	[1 1 1 2 2]	(3, 4, 3)
$\mathcal{O}_{14}$	5	[1 1 2 1 2]	(3, 3, 4)
$\mathcal{O}_{15}$	6	[1 1 1 1 1 2]	(5, 5, 5)
$\mathcal{O}_{16}$	7	[1 1 1 1 1 1 1]	(7, 7, 7)

In conclusion, any occurrence of  $a \in M$  contributes the same element to both multisets  $r(M)$  and  $r(-M)$ . Thus  $r(M) = r(-M)$ . The proof for odd  $n$  is similar.  $\square$

**Definitions 5.3.** The Short Frequency Vector (SFV) of  $r(M)$  is the vector

$$(f_{r(M)}(1), f_{r(M)}(2), \dots, f_{r(M)}(\lfloor \frac{n}{2} \rfloor)) \text{ of length } \lfloor \frac{n}{2} \rfloor.$$

**Remark 5.4.** From Lemma 4.5 we have  $\text{RIV}(\mathcal{O}(\alpha(S))) = \text{SFV}(r(\delta(S)))$ .

**Example 5.5.**  $(n, k) = (15, 6)$ . See Example 4.6. Here  $X = \{0, 4, 7, 9, 10, 14\} \subseteq \mathbb{Z}_{15}$  and  $\delta(X) = \{1^1, 2^1, 3^2, 4^2, 5^2, 6^1, 7^2, 9^1, 10^2, 14^1\}$ , and  $r(\delta(X)) = \{1^2, 2^1, 3^2, 4^2, 5^4, 6^2, 7^2\}$ . So  $\text{RIV}(\mathcal{O}(\alpha(X))) = \text{SFV}(r(\delta(X))) = (2, 1, 2, 2, 4, 2, 2)$ .

**Lemma 5.6.** Let  $S \subseteq \mathbb{Z}_n$ . Then  $\text{SFV}(r(\Delta(S))) = 2 \times \text{SFV}(r(\delta(S)))$ .

**Proof.** Now  $\Delta(S) = \delta(S) \cup -\delta(S)$ , and so  $r(\Delta(S)) = r(\delta(S)) \cup -r(\delta(S)) = r(\delta(S)) \cup r(\delta(S))$  using Lemma 5.2. Hence for any  $a \in r(\delta(S))$  we have  $f_{r(\Delta(S))}(a) = 2 \times f_{r(\delta(S))}(a)$ , and so the result.  $\square$

**Example 5.7.**  $(n, k) = (15, 6)$ . See Example 5.5. Again,  $X = \{0, 4, 7, 9, 10, 14\} \subseteq \mathbb{Z}_{15}$  and  $\Delta(X) = \{1^2, 2^2, 3^4, 4^4, 5^4, 6^2, 7^4, 9^2, 10^4, 14^2\}$ , and  $r(\Delta(X)) = \{1^4, 2^2, 3^4, 4^4, 5^8, 6^4, 7^4\}$ . So  $\text{SFV}(r(\Delta(X))) = (4, 2, 4, 4, 8, 4, 4) = 2 \times (2, 1, 2, 2, 4, 2, 2) = 2 \times \text{SFV}(r(\delta(X)))$ .

**Table 3**  
All  $(n, k)$ -Ovals up to congruence and their RIV's for  $8 \leq n \leq 10$ .

$\mathcal{O}_i$	$k$	TAIS	RIV
<b><math>n = 8</math></b>			
$\mathcal{O}_1$	2	[1 7]	(1, 0, 0, 0)
$\mathcal{O}_2$	2	[2 6]	(0, 1, 0, 0)
$\mathcal{O}_3$	2	[3 5]	(0, 0, 1, 0)
$\mathcal{O}_4$	2	[4 4]	(0, 0, 0, 1)
$\mathcal{O}_5$	3	[1 1 6]	(2, 1, 0, 0)
$\mathcal{O}_6$	3	[1 2 5]	(1, 1, 1, 0)
$\mathcal{O}_7$	3	[1 3 4]	(1, 0, 1, 1)
$\mathcal{O}_8$	3	[2 2 4]	(0, 2, 0, 1)
$\mathcal{O}_9$	3	[2 3 3]	(0, 1, 2, 0)
$\mathcal{O}_{10}$	4	[1 1 1 5]	(3, 2, 1, 0)
$\mathcal{O}_{11}$	4	[1 1 2 4]	(2, 2, 1, 1)
$\mathcal{O}_{12}$	4	[1 1 3 3]	(2, 1, 2, 1)
$\mathcal{O}_{13}$	4	[1 2 1 4]	(2, 1, 2, 1)
$\mathcal{O}_{14}$	4	[1 2 2 3]	(1, 2, 2, 1)
$\mathcal{O}_{15}$	4	[1 2 3 2]	(1, 2, 3, 0)
$\mathcal{O}_{16}$	4	[1 3 1 3]	(2, 0, 2, 2)
$\mathcal{O}_{17}$	4	[2 2 2 2]	(0, 4, 0, 2)
$\mathcal{O}_{18}$	5	[1 1 1 1 4]	(4, 3, 2, 1)
$\mathcal{O}_{19}$	5	[1 1 1 2 3]	(3, 3, 3, 1)
$\mathcal{O}_{20}$	5	[1 1 2 1 3]	(3, 2, 3, 2)
$\mathcal{O}_{21}$	5	[1 1 2 2 2]	(2, 4, 2, 2)
$\mathcal{O}_{22}$	5	[1 2 1 2 2]	(2, 3, 4, 1)
$\mathcal{O}_{23}$	6	[1 1 1 1 1 3]	(5, 4, 4, 2)
$\mathcal{O}_{24}$	6	[1 1 1 1 2 2]	(4, 5, 4, 2)
$\mathcal{O}_{25}$	6	[1 1 1 2 1 2]	(4, 4, 5, 2)
$\mathcal{O}_{26}$	6	[1 1 2 1 1 2]	(4, 4, 4, 3)
$\mathcal{O}_{27}$	7	[1 1 1 1 1 1 2]	(6, 6, 6, 3)
$\mathcal{O}_{28}$	8	[1 1 1 1 1 1 1]	(8, 8, 8, 4)
<b><math>n = 9</math></b>			
$\mathcal{O}_1$	2	[1 8]	(1, 0, 0, 0)
$\mathcal{O}_2$	2	[2 7]	(0, 1, 0, 0)
$\mathcal{O}_3$	2	[3 6]	(0, 0, 1, 0)
$\mathcal{O}_4$	2	[4 5]	(0, 0, 0, 1)
$\mathcal{O}_5$	3	[1 1 7]	(2, 1, 0, 0)
$\mathcal{O}_6$	3	[1 2 6]	(1, 1, 1, 0)
$\mathcal{O}_7$	3	[1 3 5]	(1, 0, 1, 1)
$\mathcal{O}_8$	3	[1 4 4]	(1, 0, 0, 2)
$\mathcal{O}_9$	3	[2 2 5]	(0, 2, 0, 1)
$\mathcal{O}_{10}$	3	[2 3 4]	(0, 1, 1, 1)
$\mathcal{O}_{11}$	3	[3 3 3]	(0, 0, 3, 0)
$\mathcal{O}_{12}$	4	[1 1 1 6]	(3, 2, 1, 0)
$\mathcal{O}_{13}$	4	[1 1 2 5]	(2, 2, 1, 1)
$\mathcal{O}_{14}$	4	[1 1 3 4]	(2, 1, 1, 2)
$\mathcal{O}_{15}$	4	[1 2 1 5]	(2, 1, 2, 1)
$\mathcal{O}_{16}$	4	[1 2 2 4]	(1, 2, 1, 2)
$\mathcal{O}_{17}$	4	[1 2 3 3]	(1, 1, 3, 1)
$\mathcal{O}_{18}$	4	[1 2 4 2]	(1, 2, 2, 1)
$\mathcal{O}_{19}$	4	[1 3 1 4]	(2, 0, 1, 3)
$\mathcal{O}_{20}$	4	[1 3 2 3]	(1, 1, 2, 2)
$\mathcal{O}_{21}$	4	[2 2 2 3]	(0, 3, 1, 2)
$\mathcal{O}_{22}$	5	[1 1 1 1 5]	(4, 3, 2, 1)
$\mathcal{O}_{23}$	5	[1 1 1 2 4]	(3, 3, 2, 2)
$\mathcal{O}_{24}$	5	[1 1 1 3 3]	(3, 2, 3, 2)
$\mathcal{O}_{25}$	5	[1 1 2 1 4]	(3, 2, 2, 3)
$\mathcal{O}_{26}$	5	[1 1 2 2 3]	(2, 3, 2, 3)
$\mathcal{O}_{27}$	5	[1 1 2 3 2]	(2, 3, 3, 2)
$\mathcal{O}_{28}$	5	[1 1 3 1 3]	(3, 1, 2, 4)
$\mathcal{O}_{29}$	5	[1 2 1 2 3]	(2, 2, 4, 2)
$\mathcal{O}_{30}$	5	[1 2 2 1 3]	(2, 2, 3, 3)
$\mathcal{O}_{31}$	5	[1 2 2 2 2]	(1, 4, 2, 3)

(continued on next page)

Table 3 (continued)

$\mathcal{O}_i$	$k$	TAIS	RIV
$\mathcal{O}_{32}$	6	[1 1 1 1 1 4]	(5, 4, 3, 3)
$\mathcal{O}_{33}$	6	[1 1 1 1 2 3]	(4, 4, 4, 3)
$\mathcal{O}_{34}$	6	[1 1 1 2 1 3]	(4, 3, 4, 4)
$\mathcal{O}_{35}$	6	[1 1 1 2 2 2]	(3, 5, 3, 4)
$\mathcal{O}_{36}$	6	[1 1 2 1 1 3]	(4, 3, 3, 5)
$\mathcal{O}_{37}$	6	[1 1 2 1 2 2]	(3, 4, 4, 4)
$\mathcal{O}_{38}$	6	[1 2 1 2 1 2]	(3, 3, 6, 3)
$\mathcal{O}_{39}$	7	[1 1 1 1 1 1 3]	(6, 5, 5, 5)
$\mathcal{O}_{40}$	7	[1 1 1 1 1 2 2]	(5, 6, 5, 5)
$\mathcal{O}_{41}$	7	[1 1 1 1 2 1 2]	(5, 5, 6, 5)
$\mathcal{O}_{42}$	7	[1 1 1 2 1 1 2]	(5, 5, 5, 6)
$\mathcal{O}_{43}$	8	[1 1 1 1 1 1 1 2]	(7, 7, 7, 7)
$\mathcal{O}_{44}$	9	[1 1 1 1 1 1 1 1 1]	(9, 9, 9, 9)
<b><math>n = 10</math></b>			
$\mathcal{O}_1$	2	[19]	(1, 0, 0, 0, 0)
$\mathcal{O}_2$	2	[28]	(0, 1, 0, 0, 0)
$\mathcal{O}_3$	2	[37]	(0, 0, 1, 0, 0)
$\mathcal{O}_4$	2	[46]	(0, 0, 0, 1, 0)
$\mathcal{O}_5$	2	[55]	(0, 0, 0, 0, 1)
$\mathcal{O}_6$	3	[1 1 8]	(2, 1, 0, 0, 0)
$\mathcal{O}_7$	3	[1 2 7]	(1, 1, 1, 0, 0)
$\mathcal{O}_8$	3	[1 3 6]	(1, 0, 1, 1, 0)
$\mathcal{O}_9$	3	[1 4 5]	(1, 0, 0, 1, 1)
$\mathcal{O}_{10}$	3	[2 2 6]	(0, 2, 0, 1, 0)
$\mathcal{O}_{11}$	3	[2 3 5]	(0, 1, 1, 0, 1)
$\mathcal{O}_{12}$	3	[2 4 4]	(0, 1, 0, 2, 0)
$\mathcal{O}_{13}$	3	[3 3 4]	(0, 0, 2, 1, 0)
$\mathcal{O}_{14}$	4	[1 1 1 7]	(3, 2, 1, 0, 0)
$\mathcal{O}_{15}$	4	[1 1 2 6]	(2, 2, 1, 1, 0)
$\mathcal{O}_{16}$	4	[1 1 3 5]	(2, 1, 1, 1, 1)
$\mathcal{O}_{17}$	4	[1 1 4 4]	(2, 1, 0, 2, 1)
$\mathcal{O}_{18}$	4	[1 2 1 6]	(2, 1, 2, 1, 0)
$\mathcal{O}_{19}$	4	[1 2 2 5]	(1, 2, 1, 1, 1)
$\mathcal{O}_{20}$	4	[1 2 3 4]	(1, 1, 2, 1, 1)
$\mathcal{O}_{21}$	4	[1 2 4 3]	(1, 1, 2, 2, 0)
$\mathcal{O}_{22}$	4	[1 2 5 2]	(1, 2, 2, 0, 1)
$\mathcal{O}_{23}$	4	[1 3 1 5]	(2, 0, 1, 2, 1)
$\mathcal{O}_{24}$	4	[1 3 2 4]	(1, 1, 1, 2, 1)
$\mathcal{O}_{25}$	4	[1 3 3 3]	(1, 0, 3, 2, 0)
$\mathcal{O}_{26}$	4	[1 4 1 4]	(2, 0, 0, 2, 2)
$\mathcal{O}_{27}$	4	[2 2 2 4]	(0, 3, 0, 3, 0)
$\mathcal{O}_{28}$	4	[2 2 3 3]	(0, 2, 2, 1, 1)
$\mathcal{O}_{29}$	4	[2 3 2 3]	(0, 2, 2, 0, 2)
$\mathcal{O}_{30}$	5	[1 1 1 1 6]	(4, 3, 2, 1, 0)
$\mathcal{O}_{31}$	5	[1 1 1 2 5]	(3, 3, 2, 1, 1)
$\mathcal{O}_{32}$	5	[1 1 1 3 4]	(3, 2, 2, 2, 1)
$\mathcal{O}_{33}$	5	[1 1 2 1 5]	(3, 2, 2, 2, 1)
$\mathcal{O}_{34}$	5	[1 1 2 2 4]	(2, 3, 1, 3, 1)
$\mathcal{O}_{35}$	5	[1 1 2 3 3]	(2, 2, 3, 2, 1)
$\mathcal{O}_{36}$	5	[1 1 2 4 2]	(2, 3, 2, 3, 0)
$\mathcal{O}_{37}$	5	[1 1 3 1 4]	(3, 1, 1, 3, 2)
$\mathcal{O}_{38}$	5	[1 1 3 2 3]	(2, 2, 2, 2, 2)
$\mathcal{O}_{39}$	5	[1 2 1 2 4]	(2, 2, 3, 2, 1)
$\mathcal{O}_{40}$	5	[1 2 1 3 3]	(2, 1, 4, 3, 0)
$\mathcal{O}_{41}$	5	[1 2 2 2 3]	(1, 3, 2, 3, 1)
$\mathcal{O}_{42}$	5	[1 2 2 3 2]	(1, 3, 3, 1, 2)
$\mathcal{O}_{43}$	5	[1 2 2 1 4]	(2, 2, 2, 2, 2)
$\mathcal{O}_{44}$	5	[1 2 3 1 3]	(2, 1, 3, 3, 1)
$\mathcal{O}_{45}$	5	[2 2 2 2 2]	(0, 5, 0, 5, 0)

Table 3 (continued)

$\mathcal{O}_i$	$k$	TAIS	RIV
$\mathcal{O}_{46}$	6	[1 1 1 1 1 5]	(5, 4, 3, 2, 1)
$\mathcal{O}_{47}$	6	[1 1 1 1 2 4]	(4, 4, 3, 3, 1)
$\mathcal{O}_{48}$	6	[1 1 1 1 3 3]	(4, 3, 4, 3, 1)
$\mathcal{O}_{49}$	6	[1 1 1 2 1 4]	(4, 3, 3, 3, 2)
$\mathcal{O}_{50}$	6	[1 1 1 2 2 3]	(3, 4, 3, 3, 2)
$\mathcal{O}_{51}$	6	[1 1 1 2 3 2]	(3, 4, 4, 2, 2)
$\mathcal{O}_{52}$	6	[1 1 1 3 1 3]	(4, 2, 3, 4, 2)
$\mathcal{O}_{53}$	6	[1 1 2 1 1 4]	(4, 3, 2, 4, 2)
$\mathcal{O}_{54}$	6	[1 1 2 1 2 3]	(3, 3, 4, 3, 2)
$\mathcal{O}_{55}$	6	[1 1 2 1 3 2]	(3, 3, 4, 4, 1)
$\mathcal{O}_{56}$	6	[1 1 2 2 1 3]	(3, 3, 3, 4, 2)
$\mathcal{O}_{57}$	6	[1 1 2 2 2 2]	(2, 5, 2, 5, 1)
$\mathcal{O}_{58}$	6	[1 1 3 1 1 3]	(4, 2, 2, 4, 3)
$\mathcal{O}_{59}$	6	[1 2 1 2 1 3]	(3, 2, 5, 4, 1)
$\mathcal{O}_{60}$	6	[1 2 1 2 2 2]	(2, 4, 4, 3, 2)
$\mathcal{O}_{61}$	6	[1 2 2 1 2 2]	(2, 4, 4, 2, 3)
$\mathcal{O}_{62}$	7	[1 1 1 1 1 1 4]	(6, 5, 4, 4, 2)
$\mathcal{O}_{63}$	7	[1 1 1 1 1 2 3]	(5, 5, 5, 4, 2)
$\mathcal{O}_{64}$	7	[1 1 1 1 2 1 3]	(5, 4, 5, 5, 2)
$\mathcal{O}_{65}$	7	[1 1 1 1 2 2 2]	(4, 6, 4, 5, 2)
$\mathcal{O}_{66}$	7	[1 1 1 2 1 1 3]	(5, 4, 4, 5, 3)
$\mathcal{O}_{67}$	7	[1 1 1 2 1 2 2]	(4, 5, 5, 4, 3)
$\mathcal{O}_{68}$	7	[1 1 2 1 1 2 2]	(4, 5, 4, 6, 2)
$\mathcal{O}_{69}$	7	[1 1 2 1 2 1 2]	(4, 4, 6, 5, 2)
$\mathcal{O}_{70}$	8	[1 1 1 1 1 1 1 3]	(7, 6, 6, 6, 3)
$\mathcal{O}_{71}$	8	[1 1 1 1 1 1 2 2]	(6, 7, 6, 6, 3)
$\mathcal{O}_{72}$	8	[1 1 1 1 1 2 1 2]	(6, 6, 7, 6, 3)
$\mathcal{O}_{73}$	8	[1 1 1 1 2 1 1 2]	(6, 6, 6, 7, 3)
$\mathcal{O}_{74}$	8	[1 1 1 2 1 1 1 2]	(6, 6, 6, 6, 4)
$\mathcal{O}_{75}$	9	[1 1 1 1 1 1 1 1 2]	(8, 8, 8, 8, 4)
$\mathcal{O}_{76}$	10	[1 1 1 1 1 1 1 1 1 1]	(10, 10, 10, 10, 5)

5.1. Magic Ovals and cyclic difference sets

**Definitions 5.8.** A  $(n, k, \lambda)$ -cyclic difference set –  $(n, k, \lambda)$ -CDS – is a  $k$ -subset  $D \subseteq \mathbb{Z}_n$  with the property that  $\Delta(D)$  contains every non-zero element of  $\mathbb{Z}_n$  exactly  $\lambda$  times.

In a  $(n, k, \lambda)$ -CDS straightforward counting gives:

$$\lambda(n - 1) = k(k - 1), \tag{3}$$

this shows that  $\lambda$  is even if  $n$  is even.

**Example 5.9.**  $(n, k) = (7, 3)$ .  $D = \{0, 1, 3\}$  is a  $(7, 3, 1)$ -CDS. We have  $\delta(D) = \{1, 3, 2\}$  and  $-\delta(D) = \{-1, -3, -2\} = \{6, 4, 5\}$ , giving  $\Delta(D) = \{1^1, 2^1, 3^1, 4^1, 5^1, 6^1\}$ .

Recall that, when  $n$  is odd, there are  $n$  copies of each of the  $\lfloor \frac{n}{2} \rfloor$  distinct rhombs in  $SRI_{2n}$ , i.e.,  $RIV(\{2n\}) = (n, n, \dots, n, n)$ , and, when  $n$  is even, there are  $n$  copies of each of the  $\frac{n}{2} - 1$  non-square rhombs in  $SRI_{2n}$ , but only  $\frac{n}{2}$  copies of the square, i.e.,  $RIV(\{2n\}) = (n, n, \dots, n, \frac{n}{2})$ .

**Definitions 5.10.** A magic  $(n, k, \lambda)$ -Oval is, for odd  $n$ , a  $(n, k)$ -Oval that contains exactly  $\lambda$  copies of each of the  $\lfloor \frac{n}{2} \rfloor$  distinct rhombs of  $SRI_{2n}$ , i.e., that has  $RIV = (\lambda, \lambda, \dots, \lambda, \lambda)$ , and is, for even  $n$ , a  $(n, k)$ -Oval that contains exactly  $\lambda$  copies of each of the  $\frac{n}{2} - 1$  non-square rhombs in  $SRI_{2n}$ , but only  $\frac{\lambda}{2}$  copies of the square, i.e., that has  $RIV = (\lambda, \lambda, \dots, \lambda, \frac{\lambda}{2})$ .

The following **Theorem 5.11** is a main result, it proves equivalence of a magic  $(n, k, \lambda)$ -Oval and a  $(n, k, \lambda)$ -CDS.

**Theorem 5.11.** Let  $S = \{s_1, s_2, \dots, s_k\} \subseteq \mathbb{Z}_n$ . Then  $\mathcal{O}(\alpha(S))$  is a magic  $(n, k, \lambda)$ -Oval if and only if  $S$  is a  $(n, k, \lambda)$ -CDS. Moreover,  $\lambda$  is equal to the number of 1's in TAIS  $\alpha(S)$ .

**Proof.** Necessity: let  $\mathcal{O}(\alpha(S))$  be a magic  $(n, k, \lambda)$ -Oval.

For odd  $n$ : for each  $h = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ , there are  $\lambda$  occurrences of  $h$  in  $OIT(\alpha(S))$  so, by the proof of **Lemma 4.5**, the multiset  $\delta(S)$  contains  $\lambda$  occurrences from  $\{h, n - h\}$ . Suppose  $h$  occurs  $\lambda'$  times in  $\delta(S)$  then  $n - h$  will occur  $\lambda - \lambda'$  times in  $\delta(S)$ , so  $h$  will occur  $\lambda - \lambda'$  times in  $-\delta(S)$ . Hence  $h$  will occur exactly  $\lambda$  times in  $\Delta(S) = \delta(S) \cup -\delta(S)$ . For  $h = \lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor +$

2, . . . , n - 1, we argue in a similar way with  $h$  replaced by  $n - h$  to conclude that these  $h$  also occur  $\lambda$  times in  $\Delta(S)$ . Now  $\Delta(S)$  is the multiset of differences defined by  $S$ ; hence  $S$  is a cyclic difference set with repetition number  $\lambda$ , i.e.,  $S$  is a  $(n, k, \lambda)$ -CDS.

For even  $n$ : arguing as above each  $h \neq \frac{n}{2}$  occurs  $\lambda$  times in  $\Delta(S)$ . Also  $h = \frac{n}{2}$  occurs  $\frac{\lambda}{2}$  times in  $\text{OIT}(\alpha(S))$ , i.e.,  $\frac{\lambda}{2}$  times in  $r(\delta(S))$  and so  $\frac{\lambda}{2}$  times in  $\delta(S)$ , and thus  $\lambda$  times in  $\Delta(S)$  using Lemma 5.6. Hence, for even  $n$  also,  $S$  is a  $(n, k, \lambda)$ -CDS.

Sufficiency: let  $S = \{s_1, s_2, \dots, s_k\}$  be a  $(n, k, \lambda)$ -CDS. So, for odd  $n$ , we have  $\text{SFV}(r(\Delta(S))) = (2\lambda, 2\lambda, \dots, 2\lambda, 2\lambda)$ , and, for even  $n$ , we have  $\text{SFV}(r(\Delta(S))) = (2\lambda, 2\lambda, \dots, 2\lambda, \lambda)$ . Hence, from Lemma 5.6, for odd  $n$ , we have  $\text{SFV}(r(\delta(S))) = (\lambda, \lambda, \dots, \lambda, \lambda)$ , and, for even  $n$ , we have  $\text{SFV}(r(\delta(S))) = (\lambda, \lambda, \dots, \lambda, \frac{\lambda}{2})$ . But  $\text{RIV}(\vartheta(\alpha(S))) = \text{SFV}(r(\delta(S)))$  and so  $\vartheta(\alpha(S))$  is a magic  $(n, k, \lambda)$ -Oval.

Let  $\mu$  be the number of 1's in the TAIS  $\alpha(S) = [s_2 - s_1, s_3 - s_2, \dots, s_k - s_{k-1}, s_1 - s_k]$ . Recall that the elements in  $S = \{s_1, s_2, \dots, s_k\}$  are in increasing order and satisfy  $0 \leq s_1 < s_2 < \dots < s_k$ . There are  $\lambda$  1's in  $\Delta(S)$ ; hence there are  $\lambda$  solutions to  $s_j - s_i \equiv 1 \pmod{n}$ , where  $i, j \in \{1, 2, \dots, k\}, i \neq j$ . Now if  $s_j - s_i = 1$  or  $-(n - 1)$  then  $j = i + 1$  for  $1 \leq i \leq k - 1$ , or  $j = 1$  and  $i = k$  (respectively), and thus  $s_j - s_i$  is an element of  $\alpha(S)$ . Hence  $\mu \geq \lambda$ . Conversely, because there are  $\mu$  1's in the TAIS  $\alpha(S)$  and every element of this TAIS is also an element of  $\Delta(S)$ , then  $\mu \leq \lambda$ . Hence  $\lambda = \mu$ .  $\square$

**Example 5.12.** (a) The regular  $2n$ -gon  $\{2n\}$  has TAIS =  $\underbrace{[1 \ 1 \ \dots \ 1]}_n$ , which contains  $n$  1's. It is a magic  $(n, n, n)$ -Oval with

corresponding  $(n, n, n)$ -CDS  $D = \{0, 1, \dots, n - 1\}$ . For odd  $n$  we have  $\text{RIV}(\{2n\}) = (n, n, \dots, n, n)$ , and for even  $n$   $\text{RIV}(\{2n\}) = (n, n, \dots, n, \frac{n}{2})$ .

(b) If we remove the right-hand strip of rhombs in  $\{2n\}$  we produce a magic  $(n, n - 1, n - 2)$ -Oval  $\{2n\}'$  with TAIS =  $\underbrace{[1 \ 1 \ \dots \ 1 \ 2]}_{n-1}$ , containing  $n - 2$  1's. For odd  $n$  we have  $\text{RIV}(\{2n\}') = (n - 2, n - 2, \dots, n - 2, n - 2)$ , and, for even  $n$ ,

we have  $\text{RIV}(\{2n\}') = (n - 2, n - 2, \dots, n - 2, \frac{n-2}{2})$ . The corresponding  $(n, n - 1, n - 2)$ -CDS is  $D' = \{0, 1, \dots, n - 2\}$ . See Fig. 5 for an example with  $n = 12$ .

If we remove another strip of rhombs we obtain a  $(n, n - 2)$ -Oval but only non-integer values of  $\lambda$  result from Eq. (3), and so such an Oval is not magic.

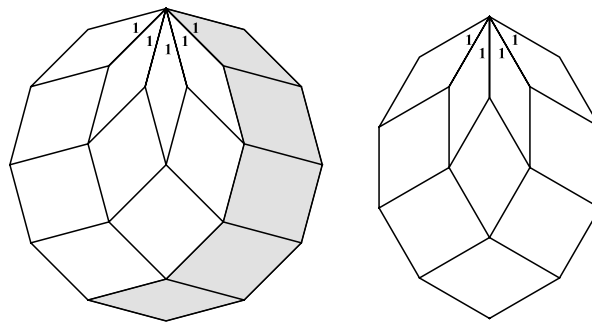


Fig. 5. The regular 12-gon  $\{12\}$ , and the magic  $(6, 5, 4)$ -Oval  $\{12\}'$  obtained by removing the right-hand strip of rhombs from  $\{12\}$ .

(c)  $(n, k) = (7, 3)$ . See Example 5.9. The set  $D = \{0, 1, 3\}$  is a  $(7, 3, 1)$ -CDS, and so  $\vartheta(\alpha(D))$  is a magic  $(7, 3, 1)$ -Oval with TAIS  $\alpha(D) = [1 \ 2 \ 4]$ , which contains one 1. The OIT for  $\vartheta(\alpha(D))$  is  $\frac{12}{3}$  and so  $\text{RIV}(\vartheta(\alpha(D))) = (1, 1, 1)$ . See the fourth  $(7, 3)$ -Oval in Fig. 4.

(d)  $(n, k) = (15, 7)$ . See Fig. 6. The set  $D = \{0, 1, 2, 4, 5, 8, 10\}$  is a  $(15, 7, 3)$ -CDS. We have  $\alpha(D) = [1 \ 1 \ 2 \ 1 \ 3 \ 2 \ 5]$ , which contains 3 1's, and the  $(15, 7)$ -Oval  $\vartheta(\alpha(D))$  is a magic  $(15, 7, 3)$ -Oval with OIT.

1	1	2	1	3	2	
	2	3	3	4	5	
		4	4	6	6	and RIV (3,3,3,3,3,3).
		5	7	7		
			7	6		
				5		



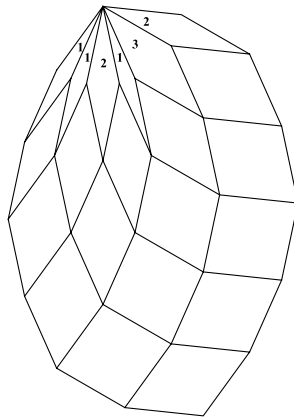


Fig. 6. The magic (15, 7, 3)-Oval  $\mathcal{O}([1\ 1\ 2\ 1\ 3\ 2\ 5])$ .

**Remark 5.13.** The CDS's  $D$  and  $D'$  in Example 5.12(a) and (b) above are usually considered to be 'trivial' CDS; see p. 298 of [3]. We ignore the other two trivial CDS, namely  $\emptyset$  and  $\{s_i\}$ , because  $k \geq 2$ . Thus non-trivial magic  $(n, k, \lambda)$ -Ovals have  $2 \leq k \leq n - 2$ .

Both these trivial CDS's have  $\text{mult}(D) = \text{mult}(D') = U(n)$ , so both have  $-1$  as a multiplier. Let  $D$  be a non-trivial  $(n, k, \lambda)$ -CDS. Then it is combinatorial folklore that  $-1$  is not a multiplier of  $D$ ; see the discussion on p. 60 of Baumert [2]. Thus  $-1$  is not a multiplier of the non-trivial magic  $(n, k, \lambda)$ -Oval  $\mathcal{O}(\alpha(D))$ . Then Theorem 3.7(ii) gives Theorem 5.14 which is a geometrical interpretation of this fact.

**Theorem 5.14.** Let  $\mathcal{O}(\alpha(D))$  be a non-trivial magic  $(n, k, \lambda)$ -Oval. Then  $-1$  is not a multiplier of  $\mathcal{O}(\alpha(D))$ , so  $\mathcal{O}(\alpha(D)) \neq \mathcal{O}(\alpha(-D))$  and  $\{\mathcal{O}(\alpha(D)), \mathcal{O}(\alpha(-D))\}$  is a congruent enantiomorphic pair in  $\mathcal{O}^*(n, k)$ .

**Example 5.15.**  $(n, k) = (7, 3)$ . See Examples 3.8 and 5.12(c). The  $(7, 3)$ -Oval  $\mathcal{O}(\alpha(\mathcal{D}))$  with  $D = \{0, 1, 3\}$  is a non-trivial magic  $(7, 3, 1)$ -Oval, so  $-1 \notin \text{mult}(\mathcal{O}(\alpha(\mathcal{D})))$  and  $\{\mathcal{O}(\alpha(\mathcal{D})), \mathcal{O}(\alpha(-\mathcal{D}))\}$  is a congruent enantiomorphic pair in  $\mathcal{O}^*(7, 3)$ .

To the end of this section we assume our CDS's are non-trivial.

**Definitions 5.16.** A  $(n, k, \lambda)$ -CDS is planar if  $\lambda = 1$ .

We now give a new proof that  $-1$  is not a multiplier of a planar CDS.

**Theorem 5.17.** Let  $D$  be a planar  $(n, k, 1)$ -CDS with  $k \geq 3$ . Then  $-1 \notin \text{mult}(D)$ .

**Proof.** Let  $T = \alpha(D) = [t_1\ t_2\ \dots\ t_k]$  be the TAIS of  $\mathcal{O}(\alpha(D))$ . Then  $\mathcal{O}(\alpha(D))$  is a magic  $(n, k, 1)$ -Oval. Suppose that two parts of  $T$  are equal, say  $t_i = t_j = h$  for  $1 \leq i < j \leq k$  and  $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$ . Now form OIT( $T$ ) using any truncated TAIS containing both  $t_i$  and  $t_j$ , this is possible because  $k \geq 3$ . Then OIT( $T$ ) will contain at least 2 copies of rhomb  $\rho_h$ , i.e.,  $\lambda_h \geq 2$  in RIV( $\mathcal{O}(\alpha(D))$ ), a contradiction because  $\lambda = \lambda_h = 1$ . So the  $k$  parts of  $T = [t_1\ t_2\ \dots\ t_k]$  are distinct.

Suppose that  $T$  is reversible, so  $T \equiv_{\text{cyc}} \overleftarrow{T}$  where  $\overleftarrow{T} = [t_k\ t_{k-1}\ \dots\ t_1]$ . Now, because the parts of  $T$  are distinct, we have  $\overleftarrow{T} \equiv_{\text{cyc}} [t_1\ t_k\ \dots\ t_2] = [t_1\ t_2\ \dots\ t_k]$ , so  $t_k = t_2$ , a contradiction. Hence  $T$  is not reversible, and, by Theorem 3.4, we have  $-1 \notin \text{mult}(D)$ .  $\square$

5.2. All magic  $(n, k, \lambda)$ -Ovals,  $n \leq 40$

See p. 2 of Baumert [2].

**Definitions 5.18.** Two  $k$ -subsets  $S$  and  $S'$  of  $\mathbb{Z}_n$  are  $(u, z)$ -equivalent,  $S \equiv_{u,z} S'$ , if there exists  $u \in U(n)$  and  $z \in \mathbb{Z}_n$  such that  $S = uS' + z$ .

Table 6.1, p. 150 of [2] contains a complete list of the 74  $(n, k, \lambda)$  triples with  $k \leq 100$  for which a  $(n, k, \lambda)$ -CDS exists, with at least one example of such a CDS for each triple.

Moreover, for the 12  $(n, k, \lambda)$  triples with  $n \leq 40$ , see our Table 4, the  $(n, k, \lambda)$ -CDS examples in Table 6.1 of [2] are all the examples up to  $(u, z)$ -equivalence. To confirm this statement for these 12 triples see Hall [5]. As a double-check for the 8 triples:  $(7, 3, 1)$ ,  $(13, 4, 1)$ ,  $(15, 7, 3)$ ,  $(19, 9, 4)$ ,  $(21, 5, 1)$ ,  $(23, 11, 5)$ ,  $(31, 6, 1)$ , and  $(37, 9, 2)$  see the explicit

**Table 4**

All non-trivial  $(n, k, \lambda)$ -CDS's (up to  $(u, z)$ -equivalence) and the corresponding TAIS's of all non-trivial magic  $(n, k, \lambda)$ -Ovals (up to congruency) for  $n \leq 40$  and  $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$ .

$(n, k, \lambda)$	$D$	TAIS
(7, 3, 1)	{0, 1, 3}	[1 2 4]
(11, 5, 2)	{0, 1, 2, 6, 9}	[1 1 4 3 2]
(13, 4, 1)	{0, 1, 3, 9}	[1 2 6 4] [1 3 2 7]
(15, 7, 3)	{0, 1, 2, 4, 5, 8, 10}	[1 1 2 1 3 2 5]
(19, 9, 4)	{0, 1, 2, 3, 5, 7, 12, 13, 16}	[1 1 1 2 2 5 1 3 3]
(21, 5, 1)	{0, 1, 6, 8, 18}	[1 5 2 10 3]
(23, 11, 5)	{0, 1, 2, 3, 5, 7, 8, 11, 12, 15, 17}	[1 1 1 2 2 1 3 1 3 2 6]
(31, 6, 1)	{0, 1, 3, 8, 12, 18}	[1 2 5 4 6 13] [1 3 6 2 5 14] [1 5 12 4 7 2] [1 7 3 2 4 14] [1 10 8 7 2 3]
(31, 15, 7)–A	{0, 1, 2, 3, 5, 7, 11, 14, 15, 16, 22, 23, 26, 28, 29}	[1 1 1 2 2 4 3 1 1 6 1 3 2 1 2] [1 1 1 3 1 2 1 6 4 1 1 2 2 3 2] [1 1 1 4 1 3 6 2 1 1 2 1 2 2 3]
(31, 15, 7)–B	{0, 1, 2, 3, 7, 9, 11, 12, 13, 18, 21, 25, 26, 28, 29}	[1 1 1 4 2 2 1 1 5 3 4 1 2 1 2]
(35, 17, 8)	{0, 1, 2, 3, 5, 6, 10, 16, 17, 18, 22, 24, 25, 27, 28, 31, 33}	[1 1 1 2 1 4 6 1 1 4 2 1 2 1 3 2 2]
(37, 9, 2)	{0, 1, 3, 7, 17, 24, 25, 29, 35}	[1 2 4 10 7 1 4 6 2] [1 3 2 4 5 2 1 7 1 2]
(40, 13, 4)	{0, 1, 2, 4, 5, 8, 13, 14, 17, 19, 24, 26, 34}	[1 1 2 1 3 5 1 3 2 5 2 8 6] [1 1 7 1 3 2 1 2 2 4 6 7 3]

examples on pp. 306–308 and p. 327 of [3]. The remaining 4 triples: (11, 5, 2), (31, 15, 7), (35, 17, 8), and (40, 13, 4) were also double-checked by the authors using computer searches and Theorem 2.9 on p. 306 of [3].

Amongst these 12 triples, for just one triple, namely (31, 15, 7), there is more than one inequivalent  $(n, k, \lambda)$ -CDS: there are two inequivalent (31, 15, 7)-CDS's, these are labeled '31A' and '31B' in Table 6.1 of [2], and 'A' and 'B' in our Table 4.

We stopped at  $n = 40$  in our Table 4 to indicate that magic  $(n, k, \lambda)$ -Ovals with  $n$  even can occur.

**Remark 5.19.** Now  $-1 \notin \text{mult}(D)$ ; hence  $\text{Mult}(D) = \text{mult}(D) \cup -\text{mult}(D)$  and  $|\text{Mult}(D)| = 2 |\text{mult}(D)|$  from Definition 3.11 and Remark 3.12.

**Example 5.20.**  $(n, k) = (13, 4)$ . The unique (13, 4, 1)-CDS up to  $(u, z)$ -equivalence is  $D = \{0, 1, 3, 9\}$ .

We have  $\text{mult}(D) = \{1, 3, 9\}$  and  $\text{Mult}(D) = \{1, 3, 4, 9, 10, 12\}$ . Now  $|U(13)| = 12$  so  $|U(13) : \text{Mult}(D)| = 2$ . A set of 2 coset representatives for  $\text{Mult}(D)$  in  $U(13)$  is  $\{1, 2\}$ . Then the 2 incongruent (13, 4, 1)-CDS's that are each  $(u, z)$ -equivalent to  $D$  are  $D$  and  $2D = \{0, 2, 5, 6\} \equiv_z \{0, 1, 8, 10\}$ , with corresponding TAIS's [1 2 6 4] and [1 3 2 7] respectively. Thus there are 2 magic (13, 4, 1)-Ovals up to congruency; see our Table 4.

A similar procedure applied to each  $(n, k, \lambda)$ -CDS of Table 6.1 of [2] for  $n \leq 40$  produces our Table 4.

**Example 5.21.**  $(n, k) = (16, 6)$ . There does not exist a (16, 6, 2)-CDS; see Example 14.20(a) on p. 425 of [3]. So there does not exist a magic (16, 6, 2)-Oval, i.e., a (16, 6)-Oval with RIV (2, 2, 2, 2, 2, 2, 1). Consider the (16, 6)-Oval  $\mathcal{O} = \mathcal{O}([1 1 2 1 5 6])$ . Then  $\text{RIV}(\mathcal{O}) = (3, 2, 2, 2, 2, 2, 1, 1)$  which is the 'closest' that the RIV with  $\lambda_8 = 1$  of a (16, 6)-Oval can be to (2, 2, 2, 2, 2, 2, 2, 1), i.e., Oval  $\mathcal{O}$  is the 'closest' that a (16, 6)-Oval with one square rhomb can be to a magic (16, 6, 2)-Oval. Oval  $\mathcal{O}$  has  $\lambda_1 = 3$  (instead of  $\lambda_1 = 2$  for a magic (16, 6, 2)-Oval), and  $\lambda_7 = 1$  (instead of  $\lambda_7 = 2$ ). Alternatively,  $S = \beta([1 1 2 1 5 6]) = \{0, 1, 2, 4, 5, 10\}$  is the 'closest' that a 6-subset  $S'$  of  $\mathbb{Z}_{16}$  with the frequency in  $\Delta(S')$  of 8 equal to 2 can be to a (16, 6, 2)-CDS. In  $\Delta(S)$  the frequencies of 1 and 15 are 3 (instead of 2), and the frequencies of 7 and 9 are 1 (instead of 2).

**6. Oval-partitions of  $\{2n\}^p$ , cyclic difference families, triangle-partitions of  $\binom{n}{2}$**

See Section 3.9 of Schoen [8] for a preliminary version of some of the research in this section; see also [9].

Let  $\mathcal{O}^p$  denote  $p$  copies of Oval  $\mathcal{O}$ , in particular  $\{2n\}^p$  denotes  $p$  copies of the regular  $2n$ -gon  $\{2n\}$ .

**Definitions 6.1.** An Oval-partition of  $\{2n\}^p$  is a partition of the rhombs from  $\{2n\}^p$  into  $q$   $(n, k_i)$ -Ovals,  $\mathcal{O}_i$ , for various  $q \geq 1$  and various  $k_i \geq 2$ :

$$\{2n\}^p \rightarrow \mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_q. \tag{4}$$

**Table 5**

All Oval-partitions ( $O$ - $p$ ) of  $\{2n\}$  and the corresponding triangle-partition ( $\Delta$ - $p$ ) of  $\binom{n}{2}$  (see Section 6.3); the values of  $\mathcal{OP}(n)$  and  $\mathcal{DO}\mathcal{P}(n)$ , for  $2 \leq n \leq 5$ . The Oval numbering  $\mathcal{O}_i$  refers to Table 2.

$n$	$\binom{n}{2}$	$q$	$O$ - $p$ of $\{2n\}$	$\Delta$ - $p$ of $\binom{n}{2}$	$\mathcal{OP}(n)$	Distinct?	$\mathcal{DO}\mathcal{P}(n)$
2	1	1	$\mathcal{O}_1$	1	1	Yes	1
3	3	1	$\mathcal{O}_2$	3	2	Yes	1
3	3	3	$\mathcal{O}_1^3$	$1^3$		No	
4	6	1	$\mathcal{O}_4$	6	4	Yes	1
4	6	2	$\mathcal{O}_3^2$	$3^2$		No	
4	6	4	$\mathcal{O}_1^2\mathcal{O}_2\mathcal{O}_3$	$1^33$		No	
4	6	6	$\mathcal{O}_1^4\mathcal{O}_2^2$	$1^6$		No	
5	10	1	$\mathcal{O}_6$	[10]	12	Yes	3
5	10	3	$\mathcal{O}_1\mathcal{O}_4\mathcal{O}_5$	136		Yes	
5	10	3	$\mathcal{O}_2\mathcal{O}_3\mathcal{O}_5$	136		Yes	
5	10	4	$\mathcal{O}_1\mathcal{O}_3\mathcal{O}_4^2$	$13^3$		No	
5	10	4	$\mathcal{O}_2\mathcal{O}_3^2\mathcal{O}_4$	$13^3$		No	
5	10	5	$\mathcal{O}_1^2\mathcal{O}_2^2\mathcal{O}_5$	$1^46$		No	
5	10	6	$\mathcal{O}_1^3\mathcal{O}_2\mathcal{O}_4^2$	$1^43^2$		No	
5	10	6	$\mathcal{O}_1^2\mathcal{O}_2^2\mathcal{O}_3\mathcal{O}_4$	$1^43^2$		No	
5	10	6	$\mathcal{O}_1\mathcal{O}_2^3\mathcal{O}_3^2$	$1^43^2$		No	
5	10	8	$\mathcal{O}_1^4\mathcal{O}_2^3\mathcal{O}_4$	$1^73$		No	
5	10	8	$\mathcal{O}_1^3\mathcal{O}_2^4\mathcal{O}_3$	$1^73$		No	
5	10	10	$\mathcal{O}_1^5\mathcal{O}_2^5$	$1^{10}$		No	

Clearly (4) is equivalent to

$$p \times \text{RIV}(\{2n\}) = \sum_{i=1}^q \text{RIV}(\mathcal{O}_i). \tag{5}$$

We focus on  $p = 1$  and sometimes shorten  $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_q$  to  $\mathcal{O}_1\mathcal{O}_2 \dots \mathcal{O}_q$ .

**Remark 6.2.** Because the regular  $2n$ -gon  $\{2n\}$  is a magic  $(n, n, n)$ -Oval then, along the lines of Theorem 5.11, we can prove that in Oval-partition (4) with  $p = 1$  the total number of 1's in the TAIS's of the Ovals in  $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_q$  equals  $n$ .

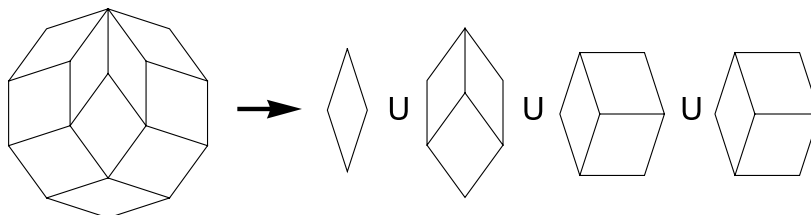
**Definitions 6.3.** Distinct Oval-partition,  $\mathcal{OP}(n)$ ,  $\mathcal{DO}\mathcal{P}(n)$ .

- (1) An Oval-partition is *distinct* if it contains distinct Ovals.
- (2)  $\mathcal{OP}(n)$  is the total number of Oval-partitions of  $\{2n\}$ , for  $n \geq 2$ ; we define  $\mathcal{OP}(1) = 1$ .
- (3)  $\mathcal{DO}\mathcal{P}(n)$  is the total number of distinct Oval-partitions of  $\{2n\}$ , for  $n \geq 2$ ; we define  $\mathcal{DO}\mathcal{P}(1) = 1$ .

See Table 5 for all Oval-partitions of  $\{2n\}$  and the corresponding triangle-partition of  $\binom{n}{2}$  (see Section 6.3), for  $n = 2, 3, 4$ , and 5.

**Example 6.4.**  $n = 5$ . See Fig. 7. As an example with  $n = 5$ , we check Eq. (5) for the Oval-partition  $\mathcal{O}_1\mathcal{O}_3\mathcal{O}_4^2$  of  $\{10\}$  from Table 5:

$$(5, 5) = (1, 0) + (2, 1) + 2(1, 2).$$



$$\{10\} \rightarrow \mathcal{O}([1\ 4]) \cup \mathcal{O}([1\ 1\ 3]) \cup \mathcal{O}([1\ 2\ 2]) \cup \mathcal{O}([1\ 2\ 2])$$

**Fig. 7.** The Oval-partition  $\mathcal{O}_1\mathcal{O}_3\mathcal{O}_4^2$  of  $\{10\}$ .

Observe that the total number of 1's in the TAIS's of the Ovals in the above Oval-partition equals  $n = 5$ , in agreement with Remark 6.2.

See Table 2,  $n = 5$ . In total there are 6  $(5, k)$ -Ovals:  $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6\}$ . Let  $\mathcal{RIV}(5) = \{\text{RIV}(\mathcal{O}_1), \text{RIV}(\mathcal{O}_2), \text{RIV}(\mathcal{O}_3), \text{RIV}(\mathcal{O}_4), \text{RIV}(\mathcal{O}_5), \text{RIV}(\mathcal{O}_6)\} = \{(1, 0), (0, 1), (2, 1), (1, 2), (3, 3), (5, 5)\}$ . Then to find all Oval-partitions of  $\{10\}$  is equivalent to finding all ways of summing elements of  $\mathcal{RIV}(5)$  to reach  $\text{RIV}(\{10\}) = (5, 5)$  where elements can be used more than once.

**Remark 6.5.** Similarly, to find all Oval-partitions of  $\{2n\}$  is equivalent to finding all ways of summing elements of the multiset of RIV's of all  $(n, k)$ -Ovals to reach  $\text{RIV}(\{2n\})$  where elements can be used more than once.

The values of  $\mathcal{OP}(n)$  and  $\mathcal{DOP}(n)$  for  $2 \leq n \leq 5$  are given in Table 5, we have also computed  $\mathcal{OP}(6) = 58$ ,  $\mathcal{DOP}(6) = 7$ ,  $\mathcal{DOP}(7) = 42$ , and  $\mathcal{DOP}(8) = 334$ . The sequences  $\{\mathcal{OP}(n) \mid n \geq 1\} = \{1, 1, 2, 4, 12, 58, \dots\}$  and  $\{\mathcal{DOP}(n) \mid n \geq 1\} = \{1, 1, 1, 1, 3, 7, 42, 334, \dots\}$  now appear in [7] as sequences A177921 and A181148 respectively.

We may also think about the Oval-partition  $\{2n\} \rightarrow \mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_q$  in terms of subsets  $S \subseteq \mathbb{Z}_n$ . From Example 5.12(a) the regular  $2n$ -gon  $\{2n\}$  is a magic  $(n, n, n)$ -Oval with corresponding  $(n, n, n)$ -CDS  $D = \{0, 1, \dots, n - 1\}$ . We modify the proof of Theorem 5.11 to give the following.

**Theorem 6.6.** The Oval-partition  $\{2n\} \rightarrow \mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_q$  exists if and only if there exists  $q$  subsets  $D_1, D_2, \dots, D_q \subseteq \mathbb{Z}_n$  with the property that  $\Delta(\{0, 1, \dots, n - 1\}) = \Delta(D_1) \cup \Delta(D_2) \cup \dots \cup \Delta(D_q)$ .

**Example 6.7.**  $n = 5$ . See Example 6.4. We have  $D = \{0, 1, 2, 3, 4\}$  and  $\Delta(D) = \{1^5, 2^5, 3^5, 4^5\}$ , and subsets of  $\mathbb{Z}_5$ :  $D_1 = \{0, 1\}$ ,  $D_2 = \{0, 1, 2\}$ , and  $D_3 = D_4 = \{0, 1, 3\}$ .

6.1. Homologous Oval-partitions, isopart triples, cyclic difference families

Here we consider Oval-partitions of  $\{2n\}^p$  in which the Ovals  $\mathcal{O}_i$  are  $(n, k)$ -Ovals, where  $k$  is fixed.

**Definitions 6.8.** A homologous Oval-partition of  $\{2n\}^p$  is a partition of the rhombs from  $\{2n\}^p$  into  $q$   $(n, k)$ -Ovals,  $\mathcal{O}_i$ , for a fixed  $k \geq 2$ :

$$\{2n\}^p \rightarrow \mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_q.$$

Note that the  $(n, k)$ -Ovals  $\mathcal{O}_i$  need not be congruent.

When  $p = 1$  for a homologous Oval-partition of  $\{2n\}$  to exist we require  $\binom{k}{2} \mid \binom{n}{2}$ . There is a homologous Oval-partition of  $\{2n\}$  into  $q = 1$   $(n, n)$ -Oval, namely into  $\{2n\}$  itself, and another into  $q = \binom{n}{2}$   $(n, 2)$ -Ovals, namely into the  $\binom{n}{2}$  rhombs of  $\{2n\}$ . We consider these two partitions as trivial, and so in the following we restrict ourselves to  $2 \leq q \leq \binom{n}{2} - 1$ .

**Definitions 6.9.**  $[(n, k), q]$  isopart triple, realizable.

(1) The ordered triple  $[(n, k), q]$  is an isopart triple if

$$\binom{n}{2} = q \binom{k}{2} \quad \text{for some } 2 \leq q \leq \binom{n}{2} - 1,$$

so  $k \geq 3$ .

(2) The isopart triple  $[(n, k), q]$  is realizable if there exists a homologous Oval-partition of  $\{2n\}$  into  $q$  (not necessarily congruent)  $(n, k)$ -Ovals.

**Example 6.10.** (a)  $[(n, k), q] = [(4, 3), 2]$ . See Table 2. The smallest isopart triple which is realizable is  $[(4, 3), 2]$ . The relevant homologous Oval-partition is  $\{8\} \rightarrow \mathcal{O}_3^2 = \mathcal{O}([1 \ 1 \ 2])^2$ .

(b)  $[(n, k), q] = [(6, 3), 5]$ . See Table 2. The smallest isopart triple which is not realizable is  $[(6, 3), 5]$ .

Suppose there is a homologous Oval-partition

$$\{12\} \rightarrow \mathcal{O}_4^{q_1} \cup \mathcal{O}_5^{q_2} \cup \mathcal{O}_6^{q_3}$$

where each  $q_i \geq 0$ . Then the system of equations containing the equation  $q_1 + q_2 + q_3 = 5$  together with the RIV Eqs. (5):

$$(6, 6, 3) = q_1(2, 1, 0) + q_2(1, 1, 1) + q_3(0, 3, 0)$$

must have a solution in the non-negative integers. That is, the system

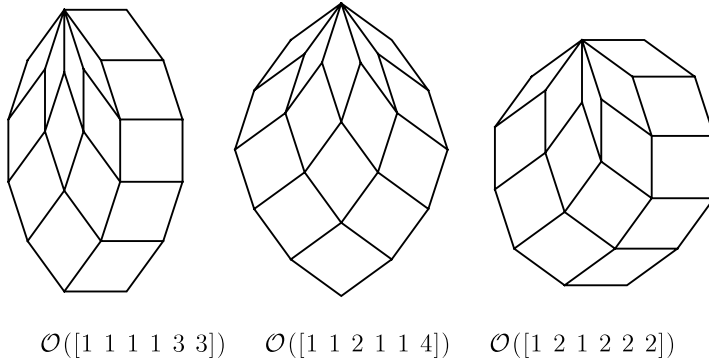
$$q_1 + q_2 + q_3 = 5, \quad 2q_1 + q_2 = 6, \quad q_1 + q_2 + 3q_3 = 6, \quad q_2 = 3,$$

must have a solution in the non-negative integers, a contradiction. Hence the isopart triple  $[(6, 3), 5]$  is not realizable.

**Table 6**

All isopart triples  $[(n, k), q]$  for  $2 \leq n \leq 16$ , and an example of a homologous Oval-partition realizing the triple. Triples  $[(6, 3), 5]$  and  $[(10, 3), 15]$  are not realizable.

$[(n, k), q]$	Example of a homologous Oval-partition realizing $[(n, k), q]$
$[(4, 3), 2]$	$\emptyset([1 1 2])^2$ (magic)
$[(6, 3), 5]$	Not realizable
$[(7, 3), 7]$	$\emptyset([1 2 4])^7$ (magic, see Table 4 row (7, 3, 1), and Example 6.19(b))
$[(9, 3), 12]$	$\emptyset([1 1 7])^3 \emptyset([1 4 4])^3 \emptyset([2 2 5])^3 \emptyset([3 3 3])^3$
$[(9, 4), 6]$	$\emptyset([1 1 2 5])^3 \emptyset([1 3 2 3])^3$
$[(10, 3), 15]$	Not realizable
$[(10, 6), 3]$	$\emptyset([1 1 1 1 3 3]) \emptyset([1 1 2 1 1 4]) \emptyset([1 2 1 2 2 2])$ (see Section 3.9 p. 22 of [8] and Fig. 8)
$[(12, 3), 22]$	$\emptyset([1 2 9])^4 \emptyset([1 3 8])^4 \emptyset([1 4 7])^4 \emptyset([2 4 6])^4 \emptyset([2 5 5])^4 \emptyset([3 3 6])^2$
$[(12, 4), 11]$	$\emptyset([1 1 3 7]) \emptyset([1 2 1 8]) \emptyset([1 2 4 5]) \emptyset([1 2 5 4]) \emptyset([1 2 2 7]) \emptyset([1 3 1 7])$ $\emptyset([1 4 1 6]) \emptyset([1 4 2 5]) \emptyset([2 2 2 6]) \emptyset([2 2 3 5]) \emptyset([3 3 3 3])$
$[(13, 3), 26]$	$\emptyset([1 3 9])^{13} \emptyset([2 5 6])^{13}$
$[(13, 4), 13]$	$\emptyset([1 2 6 4])^{13}$ (magic, see Table 4 row (13, 4, 1))
$[(15, 3), 35]$	$\emptyset([1 1 13])^5 \emptyset([1 7 7])^5 \emptyset([2 2 11])^5 \emptyset([3 3 9])^5 \emptyset([3 6 6])^5 \emptyset([4 4 7])^5 \emptyset([5 5 5])^5$
$[(15, 6), 7]$	$\emptyset([1 1 2 1 6 4]) \emptyset([1 1 2 3 2 6]) \emptyset([1 1 2 3 6 2]) \emptyset([1 2 2 7 1 2])$ $\emptyset([1 2 4 1 2 5]) \emptyset([1 2 4 1 4 3]) \emptyset([1 3 2 4 1 4])$
$[(15, 7), 5]$	$\emptyset([1 1 2 1 3 2 5])^5$ (magic, see Table 4 row (15, 7, 3), and Example 6.19(c))
$[(16, 3), 40]$	$\emptyset([1 2 13])^8 \emptyset([1 7 8])^8 \emptyset([2 4 10])^8 \emptyset([3 4 9])^8 \emptyset([5 5 6])^8$
$[(16, 4), 20]$	See Section 3.9 p. 23 of [8]
$[(16, 5), 12]$	See Example 6.11
$[(16, 6), 8]$	$\emptyset([1 1 2 1 5 6])^4 \emptyset([1 5 2 2 3 3])^4$ (see Example 6.20)



**Fig. 8.** The homologous Oval-partition of  $\{20\}$  for isopart triple  $[(10, 6), 3]$  from Table 6.

See Table 6 for all isopart triples  $[(n, k), q]$  for  $2 \leq n \leq 16$ . All are realizable except  $[(6, 3), 5]$  and  $[(10, 3), 15]$ .

**Example 6.11.**  $(n, k) = (16, 5)$ . Isopart triple  $[(16, 5), 12]$ . See Section 3.9 p. 24 of [8]. Here each of the 12  $(16, 5)$ -Ovals are distinct, i.e., incongruent. The table below gives the TAIS's and RIV's of these 12 Ovals.

TAIS	RIV
[1 1 1 3 10]	(3, 2, 2, 1, 1, 1, 0, 0)
[1 2 9 1 3]	(2, 1, 2, 2, 1, 1, 1, 0)
[1 5 2 3 5]	(1, 1, 1, 0, 3, 2, 1, 1)
[1 4 3 2 6]	(1, 1, 1, 1, 2, 1, 2, 1)
[1 2 5 1 7]	(2, 1, 1, 0, 1, 1, 2, 2)
[2 2 2 3 7]	(0, 3, 1, 2, 1, 1, 2, 0)
[2 2 3 2 7]	(0, 3, 1, 1, 2, 0, 3, 0)
[1 2 3 6 4]	(1, 1, 2, 1, 2, 2, 1, 0)
[1 3 1 3 8]	(2, 0, 2, 3, 1, 0, 1, 1)
[1 1 3 3 8]	(2, 1, 2, 1, 1, 1, 1, 1)
[2 4 2 4 4]	(0, 2, 0, 3, 0, 4, 0, 1)
[1 3 5 1 6]	(2, 0, 1, 1, 1, 2, 2, 1)
	(16, 16, 16, 16, 16, 16, 8)

Homologous Oval-partitions are closely related to another class of combinatorial objects, (cf., Theorem 6.6):

**Definitions 6.12.** A  $(n, k, \lambda)$ -cyclic difference family –  $(n, k, \lambda)$ -CDF – is a collection of  $q$   $k$ -subsets  $D_1, D_2, \dots, D_q \subseteq \mathbb{Z}_n$  with the property that  $\Delta(D_1) \cup \Delta(D_2) \cup \dots \cup \Delta(D_q)$  contains every non-zero element of  $\mathbb{Z}_n$  exactly  $\lambda$  times.

**Remark 6.13.** See Eq. (3). In a  $(n, k, \lambda)$ -CDF we have

$$\lambda(n - 1) = qk(k - 1);$$

hence  $q = \frac{\lambda(n-1)}{k(k-1)}$  is an integer.

From Definition 6.8 of a homologous Oval-partition of  $\{2n\}$  and Definition 6.12 of a  $(n, k, \lambda)$ -CDF and Theorem 6.6 we have the following result.

**Corollary 6.14.** *There exists a homologous Oval-partition of  $\{2n\}$  into  $q$   $(n, k)$ -Ovals if and only if there exists a  $(n, k, n)$ -CDF.*

Clearly, by taking unions of CDF's, there exists a  $(n, k, n)$ -CDF if and only if there exists a collection of  $(n, k, \lambda_i)$ -CDF's for which  $\sum_i \lambda_i = n$ . Hence, another main result follows.

**Theorem 6.15.** *There exists a homologous Oval-partition of  $\{2n\}$  into  $q$   $(n, k)$ -Ovals (i.e., isopart triple  $[(n, k), q]$  is realizable) if and only if there exists a collection of  $(n, k, \lambda_i)$ -CDF's for which  $\sum_i \lambda_i = n$ .*

**Example 6.16.** (a)  $(n, k) = (9, 4)$ . See Example 1.6(a) p. 470 of [3] for the  $(9, 4, 3)$ -CDF with  $D_1 = \{0, 1, 2, 4\}$  and  $D_2 = \{0, 3, 4, 7\}$ . Using 3 copies of this CDF we produce the following homologous Oval-partition of  $\{18\}$  into 6  $(9, 4)$ -Ovals:  $\mathcal{O}(\alpha(D_1))^3 \mathcal{O}(\alpha(D_2))^3 = \mathcal{O}([1\ 1\ 2\ 5])^3 \mathcal{O}([1\ 3\ 2\ 3])^3$ . This realizes isopart triple  $[(9, 4), 6]$  with the same partition as given in Table 6.

(b)  $(n, k) = (16, 3)$ . Conversely, we may take a partition which realizes an isopart triple from Table 6 and produce a CDF. For example, the 5  $(16, 3)$ -Ovals from row  $[(16, 3), 40]$ :  $\mathcal{O}([1\ 2\ 13])\mathcal{O}([1\ 7\ 8])\mathcal{O}([2\ 4\ 10])\mathcal{O}([3\ 4\ 9])\mathcal{O}([5\ 5\ 6])$  produce a  $(16, 3, 2)$ -CDF with  $D_1 = \{0, 1, 3\}$ ,  $D_2 = \{0, 1, 8\}$ ,  $D_3 = \{0, 2, 6\}$ ,  $D_4 = \{0, 3, 7\}$ , and  $D_5 = \{0, 5, 10\}$  which is not  $(u, z)$ -equivalent to the  $(16, 3, 2)$ -CDF in Examples 16.13, p. 394 of Colbourn and Dinitz [4].

(c)  $(n, k) = (6, 3)$ . From Table 6 we see that isopart triple  $[(6, 3), 5]$  is not realizable, so, from Theorem 6.15, there does not exist a  $(6, 3, 6)$ -CDF nor a  $(6, 3, 2)$ -CDF; see Table II.2.29, p. 61 of [4].

(d)  $(n, k) = (10, 3)$ . Similarly, isopart triple  $[(10, 3), 15]$  is not realizable, so there does not exist a  $(10, 3, 10)$ -CDF nor a  $(10, 3, 2)$ -CDF; see Table II.2.29, p. 61 of [4] again.

### 6.2. Magic Oval-partitions

Recall that in a  $(n, k, \lambda)$ -CDS we have  $\lambda(n - 1) = k(k - 1)$ .

As mentioned in Section 1 this research was partially motivated by Question (iii) on p. 10 of Schoen [8].

Fix  $n \geq 2$ , for which integers  $p$  and  $q$  can the rhombs contained in  $p$  copies of  $\{2n\}$  be partitioned to tile  $q$  congruent Ovals?

**Definitions 6.17.** A magic Oval-partition of  $\{2n\}^p$  is a partition of the rhombs contained in  $\{2n\}^p$  into  $q$  congruent  $(n, k)$ -Ovals,  $\mathcal{O}$ :

$$\{2n\}^p \rightarrow \mathcal{O}^q. \tag{6}$$

We now show that if such a magic Oval-partition of  $\{2n\}^p$  exists, then  $\mathcal{O}$  is magic.

**Theorem 6.18.** *The partition  $\{2n\}^p \rightarrow \mathcal{O}^q$  exists if and only if there exists a  $(n, k, \frac{pn}{q})$ -CDS, ( $\mathcal{O}$  will then be a magic  $(n, k, \frac{pn}{q})$ -Oval).*

**Proof.** For odd  $n$ . Necessity: suppose that such a partition (6) exists. Consider  $\rho_h$ , the rhomb of  $SRI_{2n}$  with principle index  $h$ , for any fixed  $h = 1, 2, \dots, \frac{n-1}{2}$ . It appears  $pn$  times on the left in partition (6) and  $q\lambda_h$  times on the right, i.e., it appears  $\lambda_h = \frac{pn}{q}$  times in  $\mathcal{O}$ . Thus  $\lambda_h$  is independent of  $h$ , and so  $\mathcal{O}$  is a magic  $(n, k, \frac{pn}{q})$ -Oval, (for some suitable  $k$  satisfying  $k(k - 1) = \frac{pn}{q}(n - 1)$ ).

Sufficiency: conversely given a magic  $(n, k, \frac{pn}{q})$ -Oval  $\mathcal{O}$  it contains  $\frac{pn}{q}$  copies of each rhomb  $\rho_h$ . So  $\mathcal{O}^q$  contains  $pn$  copies of each  $\rho_h$ , but this is exactly the number of copies of  $\rho_h$  in  $\{2n\}^p$ .

For even  $n$ . The proof is similar to the above, but we consider the non-square rhombs  $\rho_h$  for  $h = 1, 2, \dots, \frac{n}{2} - 1$ , and the square rhomb  $\rho_{\frac{n}{2}}$  as separate cases.  $\square$

We can find a partition where  $p$  and  $q$  are the smallest by considering:

$$\frac{p}{q} = \frac{\lambda}{n} = \frac{\lambda^*}{n^*}$$

where  $\gcd(\lambda^*, n^*) = 1$ . This gives the partition:

$$\{2n\}^{\lambda^*} \rightarrow \mathcal{O}^{n^*}.$$

Any other partition with the same  $\mathcal{O}$  is a ‘multiple’ of this one.

Note that if  $\lambda^* = 1$  and  $2 \leq n^* \leq \binom{n}{2} - 1$  then  $[(n, k), n^*]$  is a realizable isopart triple.

**Example 6.19.** (a) See Example 5.12(a) and (b). Oval  $\{2n\}'$  is a magic  $(n, n - 1, n - 2)$ -Oval obtained from the regular  $2n$ -gon  $\{2n\}$  by removing its right-hand strip of rhombs. For odd  $n$  we have  $\frac{\lambda}{n} = \frac{n-2}{n} = \frac{\lambda^*}{n^*}$ , so the smallest magic Oval-partition is

$$\{2n\}^{n-2} \rightarrow \{2n\}'^n.$$

For even  $n = 2m$  the smallest magic Oval-partition is

$$\{2n\}^{m-1} \rightarrow \{2n\}'^m.$$

(b) See Example 5.12(c). Oval  $\mathcal{O}([1\ 2\ 4])$  is a magic  $(7, 3, 1)$ -Oval with RIV  $(1, 1, 1)$ . Now  $\frac{\lambda}{n} = \frac{1}{7} = \frac{\lambda^*}{n^*}$ , so we have the following magic Oval-partition

$$\{14\}^1 \rightarrow \mathcal{O}([1\ 2\ 4])^7.$$

The decomposition of  $1 \times \text{RIV}(\{14\})$  is  $1 \times (7, 7, 7) \rightarrow 7 \times (1, 1, 1)$ , and the relevant realizable isopart triple is  $[(7, 3), 7]$ ; see Table 6.

(c)  $(n, k) = (15, 7)$ . See Example 5.12(d). Oval  $\mathcal{O}([1\ 1\ 2\ 1\ 3\ 2\ 5])$  is a magic  $(15, 7, 3)$ -Oval. Here  $\frac{\lambda}{n} = \frac{3}{15} = \frac{1}{5}$  so  $\lambda^* = 1$  and  $n^* = 5$ , this gives

$$\{30\}^1 \rightarrow \mathcal{O}([1\ 1\ 2\ 1\ 3\ 2\ 5])^5.$$

The RIV’s decomposition is  $1 \times (15, 15, 15, 15, 15, 15, 15) \rightarrow 5 \times (3, 3, 3, 3, 3, 3, 3)$  and  $[(15, 7), 5]$  is the corresponding realizable isopart triple.

(d)  $(n, k) = (11, 5)$ . The  $(11, 5)$ -Oval  $\mathcal{O}([1\ 1\ 4\ 3\ 2])$  is a magic  $(11, 5, 2)$ -Oval. Here  $\frac{\lambda}{n} = \frac{2}{11}$  so  $\lambda^* = 2$  and  $n^* = 11$ . This gives us a magic Oval-partition where  $p \neq 1$ :

$$\{22\}^2 \rightarrow \mathcal{O}([1\ 1\ 4\ 3\ 2])^{11}.$$

The RIV’s decomposition is  $2 \times (11, 11, 11, 11, 11) \rightarrow 11 \times (2, 2, 2, 2, 2)$ .

**Example 6.20.**  $(n, k) = (16, 6)$ . From Example 5.21 there does not exist a magic  $(16, 6, 2)$ -Oval, i.e., there does not exist a  $(16, 6)$ -Oval with RIV  $(2, 2, 2, 2, 2, 2, 1)$ . Now  $\text{RIV}(\{16\}) = (16, 16, 16, 16, 16, 16, 8)$ , so  $\{16\} \not\rightarrow \mathcal{O}^8$  where  $\mathcal{O}$  is a fixed  $(16, 6)$ -Oval. In row  $[(16, 6), 8]$  of Table 6 we gave the homologous Oval-partition

$$\{16\} \rightarrow \mathcal{O}([1\ 1\ 2\ 1\ 5\ 6])^4 \mathcal{O}([1\ 5\ 2\ 2\ 3\ 3])^4,$$

with RIV decomposition

$$(16, 16, 16, 16, 16, 16, 16, 8) = 4(3, 2, 2, 2, 2, 2, 1, 1) + 4(1, 2, 2, 2, 2, 2, 3, 1).$$

We now show that for every homologous Oval-partition  $\{16\} \rightarrow \mathcal{O}_1^{q_1} \mathcal{O}_2^{q_2}$  into exactly 2 incongruent  $(16, 6)$ -Ovals  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , we have  $q_1 = q_2 = 4$ .

Suppose  $q_1 = 1$  and  $q_2 = 7$ . Let  $\text{RIV}(\mathcal{O}_1) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$  and  $\text{RIV}(\mathcal{O}_2) = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8)$ . Then

$$(16, 16, 16, 16, 16, 16, 16, 8) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) + 7(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8),$$

and  $\lambda_h + 7\mu_h = 16$  for  $h = 1, 2, \dots, 7$ . Hence for a fixed  $h = 1, 2, \dots, 7$  we have either  $\lambda_h = \mu_h = 2$ , or  $\lambda_h = 9$  and  $\mu_h = 1$ , or  $\lambda_h = 16$  and  $\mu_h = 0$ . In particular  $\lambda_h \geq 2$  for every  $h = 1, 2, \dots, 7$ . Now  $\mathcal{O}_1$  is a  $(16, 6)$ -Oval so  $\sum_{h=1}^8 \lambda_h = \binom{6}{2} = 15$ . Thus if  $\lambda_h = 2$  for every  $h = 1, 2, \dots, 7$  then  $\lambda_8 = 1$  and  $\mathcal{O}_1$  is a magic  $(16, 6, 2)$ -Oval, a contradiction. Hence for some  $h$  with  $h = 1, 2, \dots, 7$  we must have  $\lambda_h = 9$  or  $\lambda_h = 16$ , so  $\sum_{h=1}^7 \lambda_h \geq 6 \times 2 + 9 = 21$ . But  $\sum_{h=1}^7 \lambda_h \leq 15$ , a contradiction. Hence there is no homologous Oval-partition  $\{16\} \rightarrow \mathcal{O}_1^1 \mathcal{O}_2^7$ . Similarly, the other possible homologous Oval-partitions  $\{16\} \rightarrow \mathcal{O}_1^2 \mathcal{O}_2^6$  or  $\{16\} \rightarrow \mathcal{O}_1^3 \mathcal{O}_2^5$  do not exist. Hence the only homologous Oval-partition  $\{16\} \rightarrow \mathcal{O}_1^{q_1} \mathcal{O}_2^{q_2}$  has  $q_1 = q_2 = 4$ ; an explicit example is given above.

**Table 7**

All  $\Delta$ -partitions of  $\binom{n}{2}$  for  $2 \leq n \leq 8$ . All are realizable except  $3^5$ , for  $n = 6$ .

$n$	$\binom{n}{2}$	$\Delta$ -partitions of $\binom{n}{2}$
2	1	1
3	3	3, $1^3$
4	6	6, $3^2$ , $1^3 3$ , $1^6$
5	10	[10], 136, $13^3$ , $1^4 6$ , $1^4 3^2$ , $1^7 3$ , $1^{10}$
6	15	[15], $36^2$ , $1^2 3[10]$ , $3^3 6$ , $1^3 6^2$ , $3^5$ , $1^5[10]$ , $1^3 3^2 6$ , $1^3 3^4$ , $1^6 36$ , $1^6 3^3$ , $1^9 6$ , $1^9 3^2$ , $1^{12} 3$ , $1^{15}$
7	21	[21], 6[15], $1[10]^2$ , $3^2[15]$ , $36^3$ , $1^3 3[15]$ , $1^2 3 6[10]$ , $3^3 6^2$ , $1^3 6^3$ , $1^2 3^3[10]$ , $3^5 6$ , $1^6[15]$ , $1^5 6[10]$ , $1^3 3^2 6^2$ , $3^7$ , $1^5 3^2[10]$ , $1^3 3^4 6$ , $1^6 36^2$ , $1^3 3^6$ , $1^8 3[10]$ , $1^6 3^3 6$ , $1^9 6^2$ , $1^6 3^5$ , $1^{11}[10]$ , $1^9 3^2 6$ , $1^9 3^4$ , $1^{12} 36$ , $1^{12} 3^3$ , $1^{15} 6$ , $1^{15} 3^2$ , $1^{18} 3$ , $1^{21}$
8	28	[28], 16[21], 3[10][15], $13^2[21]$ , $16^2[15]$ , $6^3[10]$ , $1^3[10][15]$ , $1^2 6[10]^2$ , $13^2 6[15]$ , $3^2 6^2[10]$ , $1^4 3[21]$ , $1^2 3^2[10]^2$ , $13^4[15]$ , $136^4$ , $3^4 6[10]$ , $1^4 36[15]$ , $1^3 36^2[10]$ , $13^3 6^3$ , $3^6[10]$ , $1^7[21]$ , $1^5 3[10]^2$ , $1^4 3^3[15]$ , $1^4 6^4$ , $1^3 3^3 6[10]$ , $13^5 6^2$ , $1^7 6[15]$ , $1^6 6^2[10]$ , $1^4 3^2 6^3$ , $1^3 3^5[10]$ , $13^7 6$ , $1^8[10]^2$ , $1^7 3^2[15]$ , $1^6 3^2 6[10]$ , $1^4 3^4 6^2$ , $13^9$ , $1^7 36^3$ , $1^6 3^4[10]$ , $1^4 3^6 6$ , $1^{10} 3[15]$ , $1^9 36[10]$ , $1^7 3^3 6^2$ , $1^4 3^8$ , $1^{10} 6^3$ , $1^9 3^3[10]$ , $1^7 3^5 6$ , $1^{13}[15]$ , $1^{12} 6[10]$ , $1^{10} 3^2 6^2$ , $1^7 3^7$ , $1^{12} 3^2[10]$ , $1^{10} 3^4 6$ , $1^{13} 36^2$ , $1^{10} 3^6$ , $1^{15} 3[10]$ , $1^{13} 3^3 6$ , $1^{16} 6^2$ , $1^{13} 3^5$ , $1^{18}[10]$ , $1^{16} 3^2 6$ , $1^{16} 3^4$ , $1^{19} 36$ , $1^{19} 3^3$ , $1^{22} 6$ , $1^{22} 3^2$ , $1^{25} 3$ , $1^{28}$

**Table 8**

All  $\Delta$ -partitions ( $\Delta$ -p) of  $\binom{6}{2} = 15$  and, except for  $3^5$ , an Oval-partition (O-p) of {12} which realizes it.

$\Delta$ -p of $\binom{6}{2}$	O-p of {12}	$\Delta$ -p of $\binom{6}{2}$	O-p of {12}	$\Delta$ -p of $\binom{6}{2}$	O-p of {12}
[15]	$\mathcal{O}_{11}$	$3^5$	Not realizable	$1^6 3^3$	$\mathcal{O}_3^3 \mathcal{O}_3^3 \mathcal{O}_4^3$
$3 6^2$	$\mathcal{O}_4 \mathcal{O}_8 \mathcal{O}_9$	$1^5[10]$	$\mathcal{O}_1^2 \mathcal{O}_2^2 \mathcal{O}_3 \mathcal{O}_{10}$	$1^9 6$	$\mathcal{O}_3^3 \mathcal{O}_4^2 \mathcal{O}_5^2 \mathcal{O}_7$
$1^2 3[10]$	$\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_5 \mathcal{O}_{10}$	$1^3 3^2 6$	$\mathcal{O}_2^2 \mathcal{O}_3^2 \mathcal{O}_4 \mathcal{O}_8$	$1^9 3^2$	$\mathcal{O}_2^2 \mathcal{O}_3^4 \mathcal{O}_4^3 \mathcal{O}_4^2$
$3^3 6$	$\mathcal{O}_4 \mathcal{O}_5^2 \mathcal{O}_8$	$1^3 3^4$	$\mathcal{O}_3^3 \mathcal{O}_4^3 \mathcal{O}_6$	$1^{12} 3$	$\mathcal{O}_4^4 \mathcal{O}_5^2 \mathcal{O}_3^3 \mathcal{O}_4$
$1^3 6^2$	$\mathcal{O}_2^2 \mathcal{O}_3 \mathcal{O}_7^2$	$1^6 3 6$	$\mathcal{O}_1 \mathcal{O}_2^2 \mathcal{O}_3^2 \mathcal{O}_4 \mathcal{O}_7$	$1^{15}$	$\mathcal{O}_1^6 \mathcal{O}_2^6 \mathcal{O}_3^3$

6.3. Triangular-partitions of  $\binom{n}{2}$

Recall the triangular numbers:  $\{\binom{n}{2}, n \geq 2\} = \{1, 3, 6, 10, 15, 21, 28, \dots\}$ .

**Definitions 6.21.** Triangular-partition ( $\Delta$ -partition) of  $\binom{n}{2}$ , realizable.

- (1) A triangular-partition ( $\Delta$ -partition) of  $\binom{n}{2}$  is an integer partition of  $\binom{n}{2}$  with each part a triangular number.
- (2) A  $\Delta$ -partition of  $\binom{n}{2}$  with  $q$  parts in which the  $i$ -th part is  $\binom{k_i}{2}$  is realizable if there exists an Oval-partition of  $\{2n\}$  into  $q$  Ovals  $\mathcal{O}_i$  in which  $\mathcal{O}_i$  is a  $(n, k_i)$ -Oval, for each  $i = 1, 2, \dots, q$ .

**Remark 6.22.** The  $\Delta$ -partition of  $\binom{n}{2}$  corresponding to the isopart triple  $[(n, k), q]$  is  $\left(\binom{k}{2}\right)^q$ .

Table 7 lists all  $\Delta$ -partitions of  $\binom{n}{2}$  for  $n = 2, 3, \dots, 8$ . For a fixed  $n$  the  $\Delta$ -partitions are given with increasing  $q$ , and then in lexicographic order for constant  $q$ . The  $\Delta$ -partition  $3^5$  of  $\binom{6}{2} = 15$  is the only  $\Delta$ -partition in Table 7 which is not realizable; see Example 6.10(b), and row [(6, 3), 5] of Table 6.

**Example 6.23.**  $2 \leq n \leq 6$ . See Table 5 for realizations of all  $\Delta$ -partitions of  $\binom{n}{2}$  for  $2 \leq n \leq 5$ . See Table 8 for all  $\Delta$ -partitions of  $\binom{6}{2} = 15$  and, except for  $3^5$ , an Oval-partition of {12} which realizes it. The  $\Delta$ -partition  $3^5$  is not realizable. The Oval numbering  $\mathcal{O}_i$  refers to Table 2.

We have extended our results on  $\Delta$ -partitions of  $\binom{n}{2}$  up to  $n = 10$ .

**Example 6.24.** For  $n = 2, 3, \dots, 10$  all  $\Delta$ -partitions of  $\binom{n}{2}$  are realizable except  $3^5$  for  $n = 6$  (see Examples 6.10(b) and 6.16(c)), and  $3^{15}$ ,  $3^8[21]$ ,  $3^5[10]^3$ ,  $3^3[36]$ , and  $3[21]^2$  for  $n = 10$ . The unrealizable  $\Delta$ -partitions for  $n = 10$  were shown to be unrealizable along the lines of Example 6.10(b) using MAPLE; see also Example 6.16(d).



**Table 9**

All  $(n, k)$ -Ovals up to  $u$ -equivalence for  $2 \leq n \leq 10$ . The equivalence class  $[\mathcal{O}_i]$  is denoted by  $\mathcal{O}_i$ ; see Example 7.7.

$n$	$k$	$\mathcal{O}_{c, \equiv_u}(n, k)$	$\mathcal{O}_{c, \equiv_u}^*(n, k)$
2	2	1	$\mathcal{O}_1$
3	2	1	$\mathcal{O}_1$
3	3	1	$\mathcal{O}_2$
4	2	2	$\mathcal{O}_1, \mathcal{O}_2$
4	3	1	$\mathcal{O}_3$
4	4	1	$\mathcal{O}_4$
5	2	1	$\mathcal{O}_1$
5	3	1	$\mathcal{O}_3$
5	4	1	$\mathcal{O}_5$
5	5	1	$\mathcal{O}_6$
6	2	3	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$
6	3	3	$\mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6$
6	4	3	$\mathcal{O}_7, \mathcal{O}_8, \mathcal{O}_9$
6	5	1	$\mathcal{O}_{10}$
6	6	1	$\mathcal{O}_{11}$
7	2	1	$\mathcal{O}_1$
7	3	2	$\mathcal{O}_4, \mathcal{O}_5$
7	4	2	$\mathcal{O}_8, \mathcal{O}_9$
7	5	1	$\mathcal{O}_{12}$
7	6	1	$\mathcal{O}_{15}$
7	7	1	$\mathcal{O}_{16}$
8	2	3	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4$
8	3	4	$\mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_7, \mathcal{O}_8$
8	4	6	$\mathcal{O}_{10}, \mathcal{O}_{11}, \mathcal{O}_{12}, \mathcal{O}_{13}, \mathcal{O}_{16}, \mathcal{O}_{17}$
8	5	4	$\mathcal{O}_{18}, \mathcal{O}_{19}, \mathcal{O}_{20}, \mathcal{O}_{21}$
8	6	3	$\mathcal{O}_{23}, \mathcal{O}_{24}, \mathcal{O}_{26}$
8	7	1	$\mathcal{O}_{27}$
8	8	1	$\mathcal{O}_{28}$
9	2	2	$\mathcal{O}_1, \mathcal{O}_3$
9	3	3	$\mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_{11}$
9	4	4	$\mathcal{O}_{12}, \mathcal{O}_{13}, \mathcal{O}_{15}, \mathcal{O}_{17}$
9	5	4	$\mathcal{O}_{22}, \mathcal{O}_{23}, \mathcal{O}_{24}, \mathcal{O}_{29}$
9	6	3	$\mathcal{O}_{32}, \mathcal{O}_{33}, \mathcal{O}_{38}$
9	7	2	$\mathcal{O}_{39}, \mathcal{O}_{41}$
9	8	1	$\mathcal{O}_{43}$
9	9	1	$\mathcal{O}_{44}$
10	2	3	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_5$
10	3	4	$\mathcal{O}_6, \mathcal{O}_7, \mathcal{O}_9, \mathcal{O}_{10}$
10	4	9	$\mathcal{O}_{14}, \mathcal{O}_{15}, \mathcal{O}_{16}, \mathcal{O}_{17}, \mathcal{O}_{18}, \mathcal{O}_{19}, \mathcal{O}_{22}, \mathcal{O}_{26}, \mathcal{O}_{27}$
10	5	9	$\mathcal{O}_{30}, \mathcal{O}_{31}, \mathcal{O}_{32}, \mathcal{O}_{33}, \mathcal{O}_{34}, \mathcal{O}_{36}, \mathcal{O}_{37}, \mathcal{O}_{38}, \mathcal{O}_{45}$
10	6	9	$\mathcal{O}_{46}, \mathcal{O}_{47}, \mathcal{O}_{48}, \mathcal{O}_{49}, \mathcal{O}_{50}, \mathcal{O}_{51}, \mathcal{O}_{53}, \mathcal{O}_{57}, \mathcal{O}_{58}$
10	7	4	$\mathcal{O}_{62}, \mathcal{O}_{63}, \mathcal{O}_{65}, \mathcal{O}_{66}$
10	8	3	$\mathcal{O}_{70}, \mathcal{O}_{71}, \mathcal{O}_{74}$
10	9	1	$\mathcal{O}_{75}$
10	10	1	$\mathcal{O}_{76}$

**7.  $u$ -equivalent Ovals**

In this section we explain why 2 incongruent  $(n, k)$ -Ovals can have RIV's that are permutations of each other. For example, see Table 2  $n = 7$ , there are 4 incongruent  $(7, 3)$ -Ovals:  $\{\mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_7\}$ , but 3 of them:  $\{\mathcal{O}_4, \mathcal{O}_6, \mathcal{O}_7\}$  have RIV's that are permutations of  $(2, 1, 0)$ .

Recall that  $S = \{s_1, s_2, \dots, s_k\}$  where  $0 \leq s_1 < s_2 < \dots < s_k$  is a  $k$ -subset of  $\mathbb{Z}_n$  with elements in increasing order. For  $u \in U(n)$ , when we form  $uS = \{us_1, us_2, \dots, us_k\}$  we will always rearrange the elements of  $uS$  in increasing order also, so that we may apply  $\alpha$  to  $uS$ . Recall also the operations  $\alpha$  and  $\beta$  from Definition 2.8, and the function  $r$  from Eq. (2). Further, we let  $\lfloor \lfloor \frac{n}{2} \rfloor \rfloor = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ .

**Lemma 7.1.** *Let principal index  $h$  occur  $\lambda_h$  times in  $\text{OIT}(\alpha(S)) = r(\delta(S))$ . Then for any  $u \in U(n)$  principal index  $uh$  occurs  $\lambda_h$  times in  $\text{OIT}(\alpha(uS)) = r(\delta(uS))$ .*

**Proof.** Let principal index  $uh$  occur  $\lambda_{uh}$  times in  $\text{OIT}(\alpha(uS)) = r(\delta(uS))$ . We must show that  $\lambda_h = \lambda_{uh}$ .

First we show  $\lambda_h \leq \lambda_{uh}$ : principal index  $h$  occurs  $\lambda_h$  times in  $\text{OIT}(\alpha(S)) = r(\delta(S))$ , so there are  $\lambda_h$  pairs  $\{s_j, s_i\}$  where  $1 \leq i < j \leq k$  for which  $s_j - s_i \in \{h, -h\}$ . Consider  $uS = \{us_1, us_2, \dots, us_k\} = \{v_1, v_2, \dots, v_k\}$  where  $0 \leq v_1 < v_2 < \dots < v_k$ . Suppose pair  $\{s_j, s_i\}$  satisfies  $s_j - s_i \in \{h, -h\}$  with  $s_j - s_i = h$ . Then  $us_j - us_i = uh$ , i.e.,  $v_\ell - v_{\ell'} = uh$  where  $v_\ell = us_j$  and  $v_{\ell'} = us_i$ . If  $\ell > \ell'$  then pair  $\{v_\ell, v_{\ell'}\}$  satisfies  $v_\ell - v_{\ell'} = uh$  and so  $v_\ell - v_{\ell'} \in \{uh, -uh\}$  and  $1 \leq \ell' < \ell \leq k$ , and if  $\ell < \ell'$  then pair  $\{v_{\ell'}, v_\ell\}$  satisfies  $v_{\ell'} - v_\ell = -uh$  and so again  $v_{\ell'} - v_\ell \in \{uh, -uh\}$  and

$1 \leq \ell < \ell' \leq k$ . Thus, in either case, a pair  $\{s_j, s_i\}$  for which  $s_j - s_i = h$  where  $1 \leq i < j \leq k$  gives rise to a pair  $\{v_a, v_b\}$  for which  $v_a - v_b \in \{uh, -uh\}$  and  $1 \leq a < b \leq k$ . Similarly if  $s_j - s_i = -h$ . Thus  $\lambda_h \leq \lambda_{uh}$ .

To show that  $\lambda_h \geq \lambda_{uh}$ , i.e.,  $\lambda_{uh} \leq \lambda_h$  we start with  $V = uS = \{us_1, us_2, \dots, us_k\} = \{v_1, v_2, \dots, v_k\}$  and argue as above with  $u$  replaced by  $u^{-1}$ .

The above two paragraphs give  $\lambda_h = \lambda_{uh}$  as required.  $\square$

**Definitions 7.2.**  $u\mathcal{O}$ , permutation  $P_u$ .

Let  $\mathcal{O}$  be a  $(n, k)$ -Oval with TAIS  $T$ , and let  $u \in U(n)$ .

(1)  $u\mathcal{O}$  is the  $(n, k)$ -Oval with TAIS  $\alpha(u\beta(T))$ .

(2) Permutation  $P_u$  is the permutation of  $[\lfloor \frac{n}{2} \rfloor]$  given by  $P_u(h) = r(uh)$ , for every  $h \in [\lfloor \frac{n}{2} \rfloor]$  and  $u \in U(n)$ .

**Theorem 7.3.** Let  $\mathcal{O}$  be a  $(n, k)$ -Oval and let  $u \in U(n)$ . Then  $RIV(u\mathcal{O}) = P_u(RIV(\mathcal{O}))$ .

**Proof.** For each  $h \in [\lfloor \frac{n}{2} \rfloor]$  let the  $h$ -th entry of  $RIV(\mathcal{O})$  be  $\lambda_h$  then, from Lemma 7.1, the  $uh$ -th entry of  $RIV(u\mathcal{O})$  is also  $\lambda_h$ . Hence  $RIV(u\mathcal{O})$  is a permutation of  $RIV(\mathcal{O})$  where, for each  $h \in [\lfloor \frac{n}{2} \rfloor]$ , the  $h$ -th entry (of  $RIV(\mathcal{O})$ ) is moved to the  $uh$ -th entry (of  $RIV(u\mathcal{O})$ ), i.e., is moved by the application of permutation  $P_u$ . Thus the result.  $\square$

**Example 7.4.** (a) For every  $n \geq 2$  we have  $-1 \in U(n)$  and  $P_{-1}$  is the identity permutation of  $[\lfloor \frac{n}{2} \rfloor]$ . Hence  $RIV(-\mathcal{O}) =$

$RIV(\mathcal{O})$ . Confirming this, see Lemma 3.2(i), we have  $TAIS(-\mathcal{O}) \equiv_{cyc} TAIS(\mathcal{O})$  and hence  $RIV(-\mathcal{O}) = RIV(\mathcal{O})$ .

(b)  $(n, k) = (15, 6)$ . See Example 2.5. For the  $(15, 6)$ -Oval  $\mathcal{X}$  with TAIS  $T = [4\ 3\ 2\ 1\ 4\ 1]$  we have  $X = \beta(T) = \{0, 4, 7, 9, 10, 14\}$ . Unit  $2 \in U(15)$  gives permutation  $P_2 = (1\ 2\ 4\ 7)(3\ 6)(5)$  of  $[7]$ . Now  $2X = \{0, 3, 5, 8, 13, 14\}$ , and so  $2\mathcal{X} = \mathcal{O}([3\ 2\ 3\ 5\ 1\ 1])$ . We check:  $RIV(2\mathcal{X}) = P_2(RIV(\mathcal{X})) = P_2(2, 1, 2, 2, 4, 2, 2) = (2, 2, 2, 1, 4, 2, 2)$ , as required by Theorem 7.3.

(c)  $(n, k) = (16, 6)$ . We show how we used Theorem 7.3 in Example 6.20. In Example 6.20 it was required to find 2  $(16, 6)$ -Ovals  $\mathcal{O}_1$  and  $\mathcal{O}_2$  for which  $RIV(\mathcal{O}_1) + RIV(\mathcal{O}_2) = (4, 4, 4, 4, 4, 4, 2)$ . From Example 5.21 we had a  $(16, 6)$ -Oval  $\mathcal{O} = \mathcal{O}([1\ 1\ 2\ 1\ 5\ 6])$  with  $RIV(\mathcal{O}) = (3, 2, 2, 2, 2, 2, 1, 1)$ . We observed that  $(4, 4, 4, 4, 4, 4, 2) - RIV(\mathcal{O}) = (1, 2, 2, 2, 2, 2, 3, 1)$  is a permutation of  $RIV(\mathcal{O})$ . Further, unit  $7 \in U(16)$  gives permutation  $P_7 = (1\ 7)(3\ 5)(2)(4)(6)(8)$  of  $[8]$ , and  $P_7(RIV(\mathcal{O})) = (1, 2, 2, 2, 2, 2, 3, 1)$ . Then letting  $\mathcal{O}_1 = \mathcal{O}$  and  $\mathcal{O}_2 = 7\mathcal{O} = \mathcal{O}([1\ 5\ 2\ 2\ 3\ 3])$  gave the required Ovals.

**Definitions 7.5.** Two  $(n, k)$ -Ovals  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are  $u$ -equivalent,  $\mathcal{O}_1 \equiv_u \mathcal{O}_2$ , if there is a  $u \in U(n)$  such that  $\mathcal{O}_1 = u\mathcal{O}_2$ .

It is clear that  $u$ -equivalence is an equivalence relation on  $\mathcal{O}_c^*(n, k)$ , the set of  $(n, k)$ -Ovals up to congruency.

**Definitions 7.6.**  $\mathcal{O}_{c, \equiv_u}^*(n, k)$ ,  $\mathcal{O}_{c, \equiv_u}(n, k)$ .

(1)  $\mathcal{O}_{c, \equiv_u}^*(n, k)$  is the set of equivalence classes of  $\equiv_u$  in  $\mathcal{O}_c^*(n, k)$ .

(2)  $\mathcal{O}_{c, \equiv_u}(n, k) = |\mathcal{O}_{c, \equiv_u}^*(n, k)|$  is the number of equivalence classes of  $\equiv_u$  in  $\mathcal{O}_c^*(n, k)$ .

**Example 7.7.**  $(n, k) = (7, 3)$ . See Table 2,  $n = 7$ . Here  $\mathcal{O}_4 = 2\mathcal{O}_6 = 4\mathcal{O}_7$ , and  $\mathcal{O}_5 = u\mathcal{O}_5$  for every  $u \in U(7)$ . Hence there are  $\mathcal{O}_{c, \equiv_u}(7, 3) = 2 \equiv_u$ -equivalence classes in  $\mathcal{O}_c^*(7, 3)$ , namely  $[\mathcal{O}_4] = \{\mathcal{O}_4, \mathcal{O}_6, \mathcal{O}_7\}$  and  $[\mathcal{O}_5] = \{\mathcal{O}_5\}$ . We have  $\mathcal{O}_{c, \equiv_u}^*(7, 3) = \{[\mathcal{O}_4], [\mathcal{O}_5]\}$ . We say that there are 2  $(7, 3)$ -Ovals up to  $u$ -equivalence, namely Ovals  $\mathcal{O}_4$  and  $\mathcal{O}_5$ ; see Table 9.

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