Let p be a prime and set $p^* = (p-1)/2$. The object is to show that

$$\{\sin\left(\frac{\pi k}{p}\right) : k = 1, \dots , p^*\}$$

is independent over $\mathbb{Q}$.

STEP 1: $f(x) = x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1$ is irreducible over $\mathbb{Q}$.
Idea: Replace x by $x+1$ and apply Eisenstein's criterion.

Set $\omega = e^{2\pi i/(2p)} = e^{\pi i/p}$, a primitive $(2p)$th root of unity.

STEP 2: $[\mathbb{Q}(\omega) : \mathbb{Q}] = p - 1$ and a basis for $\mathbb{Q}(\omega)/\mathbb{Q}$ is $\omega, \omega^2, \dots , \omega^{p-1}$.
Idea: $f(-x)$ is irreducible by STEP 1. And

$$\begin{aligned}(\omega + 1)f(-\omega) = \omega^p - &\omega^{p-1} + \omega^{p-2} - \cdots + \omega \\ &\omega^{p-1} - \omega^{p-2} + \cdots - \omega + 1 \\ &= \omega^p + 1 = 0.\end{aligned}$$

So $f(-\omega) = 0$.

STEP 3: Proof.
Suppose

$$\sum_{k=1}^{p^*} c_k \sin(k\pi/p) = 0,$$

where the $c_k \in \mathbb{Q}$. Then we have:

$$\begin{aligned}\sum_{k=1}^{p^*} \frac{c_k}{2i}\left(\omega^k - \frac{1}{\omega^k}\right) &= 0 \\ \sum_{k=1}^{p^*} c_k(\omega^k + \omega^{p-k}) &= 0 \\ \sum_{k=1}^{p^*} c_k\omega^k + \sum_{k=p^*+1}^{p-1} c_{p-k}\omega^k &= 0,\end{aligned}$$

since $p - p^* = p^* + 1$. Since $\{\omega, \omega^2, \dots , \omega^{p-1}\}$ is independent over $\mathbb{Q}$ by STEP 2, we have each $c_k = 0$.