Theorem: $(1/2)\tan \pi/7 = \sin \pi/7 + \sin 2\pi/7 - \sin 3\pi/7$ (1)

Proof: Rearranging terms in Eq. 1 gives

$$\tan \pi/7 - 2(\sin \pi/7 + \sin 2\pi/7 - \sin 3\pi/7) = 0$$

or $$\tan \pi/7 - 2(\sin 6\pi/7 + \sin 2\pi/7 - \sin 4\pi/7) = 0. \quad (2)$$

But

$$\sin 6x = 2\sin x\,(3\cos x - 16\cos^3 x + 16\cos^5 x); \quad (3)$$

$$\sin 2x = 2\sin x\cos x; \quad (4)$$

$$\sin 4x = \sin x\,(-4\cos x + 8\cos^3 x). \quad (5)$$

Now replace $\sin 6\pi/7$, $\sin 2\pi/7$, and $\sin 4\pi/7$ in the l.h.s. of Eq. 2 by applying Eqs. 3-5:

$$\text{l.hs.} = \left(\frac{\sin\frac{\pi}{7}}{\cos\frac{\pi}{7}}\right) - 2[2\sin\tfrac{\pi}{7}(3\cos\tfrac{\pi}{7} - 16\cos^3\tfrac{\pi}{7} + 16\cos^5\tfrac{\pi}{7}) + 2\sin\tfrac{\pi}{7}\cos\tfrac{\pi}{7} - \sin\tfrac{\pi}{7}(-4\cos\tfrac{\pi}{7} + 8\cos^3\tfrac{\pi}{7})] \quad (6)$$

$$= \left(\frac{\sin\frac{\pi}{7}}{\cos\frac{\pi}{7}}\right)[1 - (12\cos^2\tfrac{\pi}{7} - 64\cos^4\tfrac{\pi}{7} + 64\cos^6\tfrac{\pi}{7} + 4\cos^2\tfrac{\pi}{7} + 8\cos^2\tfrac{\pi}{7} - 16\cos^4\tfrac{\pi}{7})] \quad (7)$$

$$= \left(\frac{\sin\frac{\pi}{7}}{\cos\frac{\pi}{7}}\right)[1 - 24\cos^2\tfrac{\pi}{7} + 80\cos^4\tfrac{\pi}{7} - 64\cos^6\tfrac{\pi}{7}] \quad (8)$$

It is well known that the polynomial of smallest degree ('minimal polynomial') of which $\cos\frac{\pi}{7}$ is a root is

$$1 - 4x - 4x^2 + 8x^3$$

Hence

$$8\cos^3\tfrac{\pi}{7} = -1 + 4\cos\tfrac{\pi}{7} + 4\cos^2\tfrac{\pi}{7} \quad (9)$$

Now square both sides of Eq. 9:

$$64\cos^6\tfrac{\pi}{7} = 1 - 8\cos\tfrac{\pi}{7} + 8\cos^2\tfrac{\pi}{7} + 32\cos^3\tfrac{\pi}{7} + 16\cos^4\tfrac{\pi}{7} \quad (10)$$

Substitute from Eq. 10 for $64\cos^6\frac{\pi}{7}$ in Eq. 8:

$$\text{l.h.s.} = \left(\frac{\sin\frac{\pi}{7}}{\cos\frac{\pi}{7}}\right)[1 - 24\cos^2\tfrac{\pi}{7} + 80\cos^4\tfrac{\pi}{7} - (1 - 8\cos\tfrac{\pi}{7} + 8\cos^2\tfrac{\pi}{7} + 32\cos^3\tfrac{\pi}{7} + 16\cos^4\tfrac{\pi}{7})] \quad (11)$$

Collect terms in Eq. 11:

$$\text{l.h.s.} = \left(\frac{\sin\frac{\pi}{7}}{\cos\frac{\pi}{7}}\right)[1 - 4\cos\tfrac{\pi}{7} - 4\cos^2\tfrac{\pi}{7} + 8\cos^3\tfrac{\pi}{7}] \quad (12)$$

From Eq. 9, the r.h.s—and therefore also the l.h.s—of Eqs. 12 and 2 are equal to zero. □