Rhomb population totals $\boldsymbol{\rho}_k$ and $\boldsymbol{\rho}_{k+1}$ in generations k and $k+1$

Theorem

Let

$$\boldsymbol{\rho}_k = (\rho_k(1),\ \rho_k(2),\ \rho_k(3)),\ \text{where}\ \rho_k(i) \in \mathbf{N}\ (i = 1,2,3;\ k = 1,2,...) \quad (1)$$

$$\boldsymbol{S} = (\sin\pi/7,\ \sin 2\pi/7,\ \sin 3\pi/7) \quad (2)$$

$$A_k = \boldsymbol{\rho}_k \cdot \boldsymbol{S}\ \ (k = 1,2,...) \quad (3)$$

$$\boldsymbol{E}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 5 \end{pmatrix} \quad (4)$$

$$\boldsymbol{\rho}_{k+1} = \boldsymbol{E}_2\, \boldsymbol{\rho}_k\ \ (k = 1,2,...) \quad (5)$$

Then

$$A_{k+1} / A_k = (1 + 2\cos\pi/7)^2\ \ (k = 1,2,...) \quad (6)$$

Proof

$$A_{k+1} = \boldsymbol{\rho}_{k+1} \cdot \boldsymbol{S} \quad (7)$$

$$= \boldsymbol{E}_2\, \boldsymbol{\rho}_k \cdot \boldsymbol{S} \quad (8)$$

$$= (2\rho_k(1) + 2\rho_k(2) + \rho_k(3),\ 2\rho_k(1) + 3\rho_k(2) + 3\rho_k(3),\ \rho_k(1) + 3\rho_k(2) + 5\rho_k(3)) \cdot \boldsymbol{S} \quad (9)$$

$$\begin{aligned} &= [2\rho_k(1) + 2\rho_k(2) + \rho_k(3)]\sin\pi/7 \\ &+ [2\rho_k(1) + 3\rho_k(2) + 3\rho_k(3)]\sin 2\pi/7 \\ &+ [\rho_k(1) + 3\rho_k(2) + 5\rho_k(3)]\sin 3\pi/7 \end{aligned} \quad (10)$$

$$\begin{aligned} &= [2\rho_k(1) + 2\rho_k(2) + \rho_k(3)]\cos 5\pi/14 \\ &+ [2\rho_k(1) + 3\rho_k(2) + 3\rho_k(3)]\cos 3\pi/14 \\ &+ [\rho_k(1) + 3\rho_k(2) + 5\rho_k(3)]\cos\pi/14 \end{aligned} \quad (11)$$

$$A_k = \boldsymbol{\rho}_k \cdot \boldsymbol{S} \quad (3)$$

$$= \rho_k(1)\sin\pi/7 + \rho_k(2)\sin 2\pi/7 + \rho_k(3)\sin 3\pi/7$$

$$= \rho_k(1)\cos 5\pi/14 + \rho_k(2)\cos 3\pi/14 + \rho_k(3)\cos\pi/14 \quad (12)$$

Then $(1 + 2\cos\pi/7)^2 A_k$

$$= [3 + 4\cos\pi/7 + 2\cos\ 2\pi/7]\, A_k$$

$$= [3 + 4\cos\pi/7 + 2\cos\ 2\pi/7][\rho_k(1)\cos 5\pi/14 + \rho_k(2)\cos 3\pi/14 + \rho_k(3)\cos\pi/14] \quad (13)$$

$$
\begin{aligned}
&= 3\rho_k(1)\cos 5\pi/14 + 3\rho_k(2)\cos 3\pi/14 + 3\rho_k(3)\cos \ \pi/14 \\
&+ 4\rho_k(1)\cos 5\pi/14 \ \cos 2\pi/14 + 4\rho_k(2)\cos 3\pi/14 \ \cos 2\pi/14 + 4\rho_k(3)\cos 2\pi/14 \ \cos\pi/14 \\
&+ 2\rho_k(1)\cos 5\pi/14 \ \cos 4\pi/14 + 2\rho_k(2)\cos 4\pi/14 \ \cos 3\pi/14 + 2\rho_k(3)\cos 4\pi/14 \ \cos\pi/14
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&= 3\rho_k(1)\cos 5\pi/14 + 3\rho_k(2)\cos 3\pi/14 + 3\rho_k(3)\cos \ \pi/14 \\
&+ 2\rho_k(1)\cos 3\pi/14 + 2\rho_k(2)[\cos 5\pi/14 + \cos\pi/14] + 2\rho_k(3)[\cos 3\pi/14 + \cos\pi/14] \\
&+ \rho_k(1)[\cos\pi/14 - \cos 5\pi/14] + \rho_k(2)\cos\pi/14 + \rho_k(3)[\cos 5\pi/14 + \cos 3\pi/14]
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&= [2\rho_k(1) + 2\rho_k(2) + \ \rho_k(3)]\cos 5\pi/14 \\
&+ [2\rho_k(1) + 3\rho_k(2) + 3\rho_k(3)]\cos 3\pi/14 \\
&+ [\ \rho_k(1) + 3\rho_k(2) + 5\rho_k(3)]\cos\pi/14
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&= [2\rho_k(1) + 2\rho_k(2) + \ \rho_k(3)]\sin\pi/7 \\
&+ [2\rho_k(1) + 3\rho_k(2) + 3\rho_k(3)]\sin 2\pi/7 \\
&+ [\ \rho_k(1) + 3\rho_k(2) + 5\rho_k(3)]\sin 3\pi/7
\end{aligned} \tag{17}
$$

$$
= A_{k+1} \tag{18}
$$

□