Expansion of blunt mirror edge length in each generation

Theorem 2:

Let

$$\boldsymbol{\sigma}_k = (\sigma_k(1), \sigma_k(2), \sigma_k(3)), \text{ where } \sigma_k(i) \in \mathbf{Z}\,(i=1,2,3;\ k=1,2,\ldots) \tag{1}$$

$$\boldsymbol{E}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{2}$$

$$\boldsymbol{\delta} = (1,1,-1) \tag{3}$$

$$\boldsymbol{\tau}_k = \boldsymbol{\sigma}_k + \boldsymbol{\delta} \tag{4}$$

$$\boldsymbol{\sigma}_{k+1} = \boldsymbol{E}_3 \boldsymbol{\tau}_k^{\mathsf{T}} - \boldsymbol{\delta} \tag{5}$$

$$\boldsymbol{S} = \left( \sin\frac{\pi}{7},\ \sin\frac{2\pi}{7},\ \sin\frac{3\pi}{7} \right) \tag{6}$$

$$\lambda_k = \boldsymbol{\tau}_k \cdot \boldsymbol{S} \quad (k=1,2,\ldots) \tag{7}$$

Then

$$\lambda_{k+1} = \left( 1 + 2\cos\frac{\pi}{7} \right) \lambda_k \quad (k=1,2,\ldots) \tag{8}$$

Proof:

From Eqs. 3, 4, and 7,

$$\lambda_k = (\sigma_k(1)+1)\sin\frac{\pi}{7} + (\sigma_k(2)+1)\sin\frac{2\pi}{7} + (\sigma_k(3)-1)\sin\frac{3\pi}{7} \tag{9}$$

and

$$\lambda_{k+1} = \boldsymbol{\tau}_{k+1} \cdot \boldsymbol{S}$$

$$= (\boldsymbol{\sigma}_{k+1} + \boldsymbol{\delta}) \cdot \boldsymbol{S} \tag{10}$$

$$= \left( \left( \boldsymbol{E}_3 \boldsymbol{\tau}_k^{\mathsf{T}} - \boldsymbol{\delta}^{\mathsf{T}} \right) + \boldsymbol{\delta}^{\mathsf{T}} \right) \cdot \boldsymbol{S} \tag{11}$$

$$= \boldsymbol{E}_3 (\boldsymbol{\sigma}_k + \boldsymbol{\delta})^{\mathsf{T}} \cdot \boldsymbol{S} \tag{12}$$

$$\begin{aligned} &= \left( \sigma_k(1) + \sigma_k(2) + 2 \right) \sin\frac{\pi}{7} \\ &+ \left( \sigma_k(1) + \sigma_k(2) + \sigma_k(3) + 1 \right) \sin\frac{2\pi}{7} \\ &+ \left( \sigma_k(2) + 2\sigma_k(3) - 1 \right) \sin\frac{3\pi}{7} \end{aligned} \tag{13}$$

$$\begin{aligned} &= \left( \sigma_k(1)+1 \right)\sin\frac{\pi}{7} + \left( \sigma_k(2)+1 \right)\sin\frac{2\pi}{7} + \left( \sigma_k(3)-1 \right)\sin\frac{3\pi}{7} \\ &+ \sigma_k(1)\sin\frac{2\pi}{7} + \sigma_k(2)\left( \sin\frac{\pi}{7} + \sin\frac{3\pi}{7} \right) + \sigma_k(3)\left( \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} \right) + \sin\frac{\pi}{7} \end{aligned} \tag{14}$$

$$
\begin{aligned}
&= \lambda_k + \sigma_k(1)\sin\frac{2\pi}{7} \\
&\quad + \sigma_k(2)\left(\sin\frac{\pi}{7} + \sin\frac{3\pi}{7}\right) \\
&\quad + \sigma_k(3)\left(\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7}\right) + \sin\frac{\pi}{7}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&= \lambda_k + \sigma_k(1)\cos\frac{3\pi}{14} \\
&\quad + \sigma_k(2)\left(\cos\frac{5\pi}{14} + \cos\frac{\pi}{14}\right) \\
&\quad + \sigma_k(3)\left(\cos\frac{3\pi}{14} + \cos\frac{\pi}{14}\right) + \cos\frac{5\pi}{14}
\end{aligned} \tag{16}
$$

But $\quad (1 + 2\cos\frac{\pi}{7})\,\lambda_k = \lambda_k + 2\lambda_k\cos\frac{\pi}{7}$ (17)

It follows from Eqs. 16 and 17 that Eq. 8 will be proved correct if it is proved that

$$
\begin{aligned}
2\lambda_k \cos\frac{\pi}{7} &= \sigma_k(1)\,\cos\frac{3\pi}{14} \\
&\quad + \sigma_k(2)\left(2\cos\frac{3\pi}{14}\,\cos\frac{2\pi}{14}\right) \\
&\quad + \sigma_k(3)\left(2\cos\frac{2\pi}{14}\,\cos\frac{\pi}{14}\right) + \cos\frac{5\pi}{14}
\end{aligned} \tag{18}
$$

Now substitute for $\lambda_k$ in the l.h.s. of Eq. 18 from Eq. 9:

$$
2\lambda_k \cos\frac{\pi}{7} = \left(2\cos\frac{\pi}{7}\right)\left(\left(\sigma_k(1)+1\right)\sin\frac{\pi}{7} + \left(\sigma_k(2)+1\right)\sin\frac{2\pi}{7} + \left(\sigma_k(3)-1\right)\sin\frac{3\pi}{7}\right) \tag{19}
$$

$$
= \left(2\cos\frac{\pi}{7}\right)\left(\left(\sigma_k(1)+1\right)\cos\frac{5\pi}{14} + \left(\sigma_k(2)+1\right)\cos\frac{3\pi}{14} + \left(\sigma_k(3)-1\right)\cos\frac{\pi}{14}\right) \tag{20}
$$

$$
\begin{aligned}
&= \sigma_k(1)\left(2\cos\frac{5\pi}{14}\,\cos\frac{2\pi}{14}\right) + 2\cos\frac{5\pi}{14}\,\cos\frac{2\pi}{14} \\
&\quad + \sigma_k(2)\left(2\cos\frac{3\pi}{14}\,\cos\frac{2\pi}{14}\right) + 2\cos\frac{3\pi}{14}\cos\frac{2\pi}{14} \\
&\quad + \sigma_k(3)\left(2\cos\frac{2\pi}{14}\,\cos\frac{\pi}{14}\right) - 2\cos\frac{2\pi}{14}\,\cos\frac{\pi}{14}
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&= \sigma_k(1)\cos\frac{3\pi}{14} + \cos\frac{3\pi}{14} \\
&\quad + \sigma_k(2)\left(2\cos\frac{3\pi}{14}\ \cos\frac{\pi}{14}\right) + 2\cos\frac{3\pi}{14}\ \cos\frac{2\pi}{14} \\
&\quad + \sigma_k(3)\left(2\cos\frac{2\pi}{14}\ \cos\frac{\pi}{14}\ \right) - 2\cos\frac{2\pi}{14}\ \cos\frac{\pi}{14}
\end{aligned}
\tag{22}
$$

In Eq. 22, replace the expression $\cos\frac{3\pi}{14} + \left(2\cos\frac{3\pi}{14}\cos\frac{2\pi}{14}\right) - \left(2\cos\frac{2\pi}{14}\cos\frac{\pi}{14}\right)$ by its equivalent, $\cos\frac{5\pi}{14}$, yielding

$$
\begin{aligned}
2\lambda_k \cos\frac{\pi}{7} &= \sigma_k(1)\cos\frac{3\pi}{14} \\
&\quad + \sigma_k(2)\left(2\cos\frac{3\pi}{14}\ \cos\frac{2\pi}{14}\right) \\
&\quad + \sigma_k(3)\left(2\cos\frac{2\pi}{14}\ \cos\frac{\pi}{14}\right) + \cos\frac{5\pi}{14}
\end{aligned}
\tag{18}
$$

□