Theorem 1.1:

If the signatures of consecutive **pointed mirrors** for odd $n \geq 5$ are related recursively via the matrix E_m,

the ratio $\lambda_{k+1} / \lambda_k$ of their edge lengths = $1 + 2\cos\frac{\pi}{n}$ $(k = 1, 2, \ldots)$.

Let

$$m = \lfloor n/2 \rfloor ; \tag{1}$$

$$\boldsymbol{\sigma}_k = (\sigma_k(1),\, \sigma_k(2),\, \ldots,\, \sigma_k(m)),$$

where $\sigma_k(i) \in \mathbf{Z}\,(k = 1, 2, \ldots;\; i = 1, 2, \ldots, m)$; (2)

$$S = \left(\sin\frac{\pi}{n},\, \sin\frac{2\pi}{n},\, \ldots,\, \sin\frac{m\pi}{n} \right); \tag{3}$$

E_m = the symmetric m x m matrix in which

each super-diagonal and sub-diagonal element = 1;

each main diagonal element = 1, except for the element $e_{m,m}$, which = 2;

every other element = 0; (4)

$$E_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},\; E_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix},\; E_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}\; (etc.)$$

$$\boldsymbol{\sigma}_{k+1} = E_m \boldsymbol{\sigma}_k^{\mathsf{T}} \;\; (k = 1, 2, \ldots); \tag{5}$$

$$\lambda_k = \boldsymbol{\sigma}_k \cdot S \quad (k = 1, 2, \ldots); \tag{6}$$

Then

$$\lambda_{k+1} = \left(1 + 2\cos\frac{\pi}{n} \right) \lambda_k \quad (k = 1, 2, \ldots) \tag{7}$$

Proof:

$$\lambda_{k+1} = \sigma_{k+1} \cdot S \tag{8}$$

$$= E_m \boldsymbol{\sigma}_k^{\mathsf{T}} \cdot S \tag{9}$$

$$= \begin{pmatrix} \sigma_k(1) + \sigma_k(2) \\ \sigma_k(1) + \sigma_k(2) + \sigma_k(3) \\ \sigma_k(2) + \sigma_k(3) + \sigma_k(4) \\ \cdots \\ \sigma_k(m-2) + \sigma_k(m-1) + \sigma_k(m) \\ \sigma_k(m-1) + 2\sigma_k(m) \end{pmatrix} \cdot S \tag{10}$$

$$\begin{aligned} &= \left[\sigma_k(1) + \sigma_k(2)\right] \sin\frac{\pi}{n} \\ &+ \left[\sigma_k(1) + \sigma_k(2) + \sigma_k(3)\right] \sin\frac{2\pi}{n} \\ &+ \left[\sigma_k(2) + \sigma_k(3) + \sigma_k(4)\right] \sin\frac{3\pi}{n} \\ &+ \qquad \cdots \\ &+ \left[\sigma_k(m-2) + \sigma_k(m-1) + \sigma_k(m)\right] \sin\frac{(m-1)\pi}{n} \\ &+ \left[\sigma_k(m-1) + 2\sigma_k(m)\right] \sin\frac{m\pi}{n} \end{aligned} \tag{11}$$

$$\begin{aligned} &= \left[\sigma_k(1) + \sigma_k(2)\right] \cos\frac{(n-2)\pi}{2n} \\ &+ \left[\sigma_k(1) + \sigma_k(2) + \sigma_k(3)\right] \cos\frac{(n-4)\pi}{2n} \\ &+ \left[\sigma_k(2) + \sigma_k(3) + \sigma_k(4)\right] \cos\frac{(n-6)\pi}{2n} \\ &+ \qquad \cdots \\ &+ \left[\sigma_k(m-2) + \sigma_k(m-1) + \sigma_k(m)\right] \cos\frac{3\pi}{2n} \\ &+ \left[\sigma_k(m-1) + 2\sigma_k(m)\right] \cos\frac{\pi}{2n} \end{aligned} \tag{12}$$

$$
\begin{aligned}
&= \sigma_k(1)\cos\frac{(n-2)\pi}{2n} + \sigma_k(2)\cos\frac{(n-4)\pi}{2n} + \sigma_k(3)\cos\frac{(n-6)\pi}{2n} + \cdots + \sigma_k(m-1)\cos\frac{3\pi}{2n} + \sigma_k(m)\cos\frac{\pi}{2n} \\
&+ \sigma_k(1)\cos\frac{(n-4)\pi}{2n} \\
&+ \sigma_k(2)\left[\cos\frac{(n-2)\pi}{2n} + \cos\frac{(n-6)\pi}{2n}\right] \\
&+ \sigma_k(3)\left[\cos\frac{(n-4)\pi}{2n} + \cos\frac{(n-8)\pi}{2n}\right] \\
&+ \qquad \cdots \\
&+ \sigma_k(m-2)\left[\cos\frac{7\pi}{2n} + \cos\frac{3\pi}{2n}\right] \\
&+ \sigma_k(m-1)\left[\cos\frac{5\pi}{2n} + \cos\frac{\pi}{2n}\right] \\
&+ \sigma_k(m)\left[\cos\frac{3\pi}{2n} + \cos\frac{\pi}{2n}\right]
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&= \sigma_k(1)\cos\frac{(n-2)\pi}{2n} + \sigma_k(2)\cos\frac{(n-4)\pi}{2n} + \sigma_k(3)\cos\frac{(n-6)\pi}{2n} + \cdots + \sigma_k(m-1)\cos\frac{3\pi}{2n} + \sigma_k(m)\cos\frac{\pi}{2n} \\
&+ \sigma_k(1)\left[2\cos\frac{(n-2)\pi}{2n}\cos\frac{\pi}{n}\right] \\
&+ \sigma_k(2)\left[2\cos\frac{(n-4)\pi}{2n}\cos\frac{\pi}{n}\right] \\
&+ \sigma_k(3)\left[2\cos\frac{(n-6)\pi}{2n}\cos\frac{\pi}{n}\right] \\
&+ \qquad \cdots \\
&+ \sigma_k(m-2)\left[2\cos\frac{5\pi}{2n}\cos\frac{\pi}{n}\right] \\
&+ \sigma_k(m-1)\left[2\cos\frac{3\pi}{2n}\cos\frac{\pi}{n}\right] \\
&+ \sigma_k(m)\left[2\cos\frac{\pi}{2n}\cos\frac{\pi}{n}\right]
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&= \sigma_k(1)\left(1+2\cos\frac{\pi}{n}\right)\cos\frac{(n-2)\pi}{2n} \\
&+ \sigma_k(2)\left(1+2\cos\frac{\pi}{n}\right)\cos\frac{(n-4)\pi}{2n} \\
&+ \sigma_k(3)\left(1+2\cos\frac{\pi}{n}\right)\cos\frac{(n-6)\pi}{2n} \\
&+ \qquad \cdots \\
&+ \sigma_k(m-2)\left(1+2\cos\frac{\pi}{n}\right)\cos\frac{5\pi}{2n} \\
&+ \sigma_k(m-1)\left(1+2\cos\frac{\pi}{n}\right)\cos\frac{3\pi}{2n} \\
&+ \sigma_k(m)\left(1+2\cos\frac{\pi}{n}\right)\cos\frac{\pi}{2n}
\end{aligned} \tag{15}
$$

$$
= \left(1+2\cos\frac{\pi}{n}\right)\left(\begin{aligned}
&\sigma_k(1)\cos\frac{(n-2)\pi}{2n} \\
&+\sigma_k(2)\cos\frac{(n-4)\pi}{2n} \\
&+\sigma_k(3)\cos\frac{(n-6)\pi}{2n} \\
&+ \qquad \cdots \\
&+\sigma_k(m-2)\cos\frac{5\pi}{2n} \\
&+\sigma_k(m-1)\cos\frac{3\pi}{2n} \\
&+\sigma_k(m)\cos\frac{\pi}{2n}
\end{aligned}\right) \tag{16}
$$

$$=\left(1+2\cos\frac{\pi}{n}\right)\left(\begin{array}{l}\sigma_k(1)\sin\frac{\pi}{n}\\ +\sigma_k(2)\sin\frac{2\pi}{n}\\ +\sigma_k(3)\sin\frac{3\pi}{n}\\ + \qquad \cdots\\ +\sigma_k(m-2)\sin\frac{(m-2)\pi}{n}\\ +\sigma_k(m-1)\sin\frac{(m-1)\pi}{n}\\ +\sigma_k(m)\sin\frac{m\pi}{n}\end{array}\right) \tag{17}$$

$$=\left(1+2\cos\frac{\pi}{n}\right)\lambda_k \tag{18}$$

□