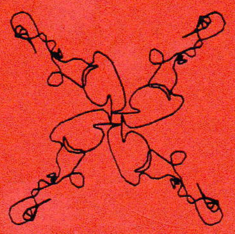
AN EFFICIENT ALGORITHM FOR LOCATING LINES OF REFLECTION IN K-PATTERNS (COMPUTER GRAPHICS PATTERNS DERIVED FROM PARTIAL SUMS OF POWER RESIDUES)



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ABSTRACT: Necessary and sufficient conditions for reflection symmetries in *K-patterns*, constructed in $\mathbf{R}^2$ by joining with line segments the consecutive points generated by partial trigonometric sums of power residues, are considered. The necessary condition for reflection symmetry is expressed by a polynomial congruence. It is proved that if a certain 'parity class conjecture' is true, then for any odd exponent, a sufficient condition for reflection symmetry is the existence of a solution of a particular linear congruence.

1. Introduction

Let us define a K-pattern as the chain of unit vectors in $\mathbf{R}^2$ obtained by joining consecutive points

$$\mathbf{r}_k = \sum_{j=0}^{k} exp[2\pi i(j_0+js)^{\alpha}/n] \qquad (1.1)$$

where

k = 0,1,2,...

j = loop index

j_0 = initial value of loop index

s = step

α = exponent

and n = modulus.

We call j_0+js the *argument* of the unit vector $exp[2\pi i(j_0+js)^{\alpha}/n]$.

We identify any particular instance of a K-pattern by its *descriptive label,* which is written as an ordered list of parameter values: K(n, s, j_0, α). Let m = the fundamental period of the K-pattern, and **sym** = a label -- described below -- for its symmetry group. We will denote the order of this group by [**sym**]. If the K-pattern is bounded, then m is defined as the number of unit vectors comprising the pattern. If the K-pattern is of frieze type (unbounded lattice pattern), then m is defined as the number of unit vectors which comprise a fundamental region of the pattern[1]. If both m and **sym** are known for a particular K-pattern, we will sometimes include them in the following *expanded* form of the descriptive label: K(n, s, j_0 , α, m, **sym**).

An obvious question is:

How do the period and symmetry of K-patterns depend on n, s, j_0, and α[2] ?

Although this question has not yet been answered completely, a number of partial answers have been obtained. The dependence of the period m on n and s when $j_0 = 0$, for any integer α, is described by a theorem which is stated in [1]. A variety of special *families* of K-patterns, for which the dependence of both period and symmetry on n, s, j_0, and α has been derived, have been described by the author[3]. For a number of other families, this dependence has been determined empirically from the study of systematically related examples. It should not be difficult to prove most or all of these latter results, using the same elementary methods which were employed for the rigorously established families.

Properly scaled and centered K-patterns can be drawn rapidly with computer graphics on a personal computer. The symmetry of the pattern can be emphasized effectively by using more than one color to draw the pattern[4]. If a multi-colored K-pattern has any reflection isometries, then in order that it display *color* reflection symmetry, it is necessary first to determine the precise location of each line of reflection with respect to the unit vectors comprising the pattern. This note describes an efficient algorithm for this determination[5].

2. The Classification of K-patterns with Reflection Symmetry

Assume that K(n, s, j_0, α) is a K-pattern which contains at least one line of reflection symmetry. If α is odd, we will refer to such a K-pattern as a K'-pattern. A K'-pattern may be either:

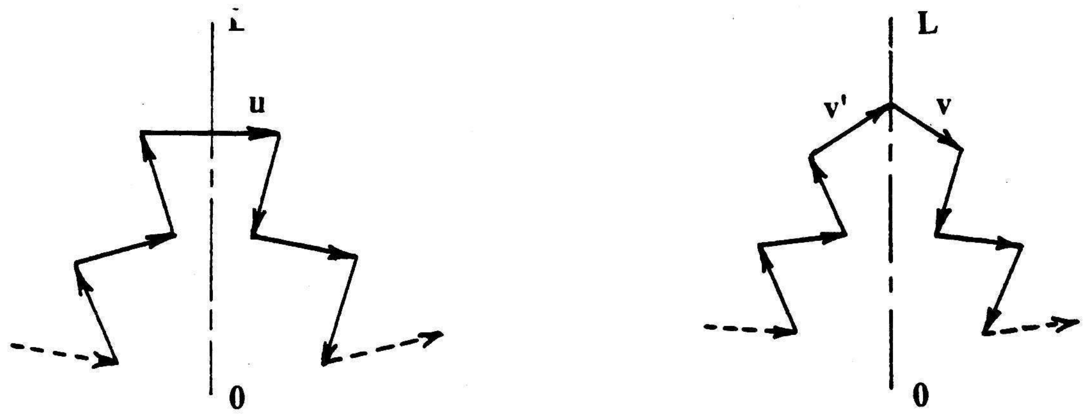
(a) bounded (**sym** = dihedral symmetry group d_i (i=1,2,...))

or (b) unbounded (**sym** = symmetry group $\mathbb{L}$, which characterizes a frieze pattern generated by two [parallel] lines of reflection[6]).

It is easy to prove that for both bounded and unbounded K'-patterns, for any sufficiently short open chain of consecutive pattern vectors which is symmetrical by reflection in a line **0L**, there are two possible types of pattern vector configuration with respect to that line:

split (S) (cf. Fig. 1a),

and *unsplit* (U) (cf. Fig. 1b).



split configuration S
(a)

unsplit configuration U
(b)

Fig. 1
The Two Types (S and U) of Pattern Vector Configuration
Associated With a Half-Line of Reflection **0L**

In Fig. 1a, the *single* [undirected] segment defined by the unit vector **u** is symmetrical by reflection in **0L**; in Fig. 1b, the *pair* of segments defined by consecutive unit vectors **v'**, **v** is symmetrical by reflection in **0L**. In case (a), we let

$$\mathbf{u} = exp(2\pi i\lambda^{\alpha}/n) \tag{2.1}$$

where

$$\lambda = j_0 + Ys \quad (Y = 0,1,2,...). \tag{2.2}$$

In case (b), we let

$$\mathbf{v} = exp(2\pi i\mu^{\alpha}/n). \tag{2.3}$$

where

$$\mu = j_0 + Zs \quad (Z = 0,1,2,...). \tag{2.4}$$

We wish to determine Y in case (a) and Z in case (b).

Let us now define four *parity classes* for bounded K'-patterns. We first define the *complexity* **comp** of any bounded K'-pattern by the relation

$$\mathbf{comp} = m\,/\,[\mathbf{sym}]; \tag{2.5}$$

m = the fundamental period, and [**sym**] = the order of the dihedral symmetry group $d_{[\mathbf{sym}]}$ of the pattern. Let us partition the set of all bounded K'-patterns into four disjoint parity classes, defined according to whether {parity ([**sym**]), parity (**comp**)} =

{e,e} (parity class 1);
{e,o} (parity class 2); (2.6)
{o,e} (parity class 3);
or {o,o} (parity class 4).

What makes these classifications significant is the following empirical observation, which we call the 'parity class conjecture':

The two types of pattern-vector configuration which are associated with any pair of consecutive half-lines of reflection symmetry are completely determined by the parity class of the pattern.

For parity classes 2 and 4, this statement is a trivial consequence of the fact that **comp** is odd, but for classes 1 and 3, where **comp** is even, it remains unproved. The distribution of pattern-vector configurations for representative K'-patterns chosen from the four parity classes is illustrated in Fig. 2.

For parity classes 2 and 4 (odd **comp**), the number f of consecutive pattern vectors which belong to a single fundamental region of the reflection group of the pattern is half-integer:

$$f = m / (2[\mathbf{sym}]) \\ = \mathbf{comp}/2. \tag{2.7}$$

(The reason for the 2 in the denominator of Eq. 2.7 is that the total number of half-lines of reflection = 2[**sym**].) From this half-integer property of f, it follows easily that the pair of pattern-vector configurations associated with two consecutive half-lines of reflection must be of 'mixed' type SU (or -- equivalently -- US).

For parity classes 1 and 3 (even **comp**), f is also given by Eq. 2.7, but f is now integer-valued. *For parity class 1, only patterns of type UU are found, whereas for parity class 3, only SS patterns occur.* Although the author has been unable to prove that no SS patterns belong to parity class 1, and that no UU patterns belong to parity class 3, the experimental evidence supporting these propositions is very strong. We will assume here that both propositions are true. They are made even more plausible by the observed behavior of frieze patterns, which we consider next.

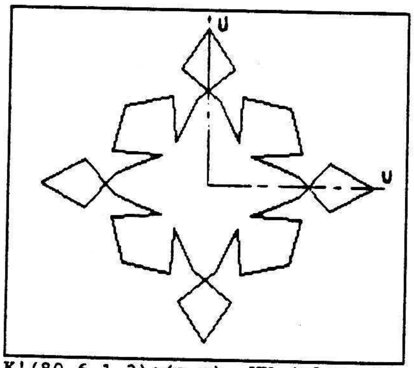
Consider the case of unbounded K'-patterns -- frieze patterns with **sym** = $\mathbb{L}$. When we defined the complexity of a bounded K'-pattern, we chose to let **comp** = m / [**sym**] (Eq. 2.5). Equivalently, we could have defined **comp** to be equal to

(a) twice the number f of consecutive pattern vectors associated with a fundamental region of the reflection group of the pattern, or

(b) the number of consecutive pattern vectors associated with a fundamental region of the rotation group of the pattern. For unbounded K'-patterns, the application of definition (a) leads to the result

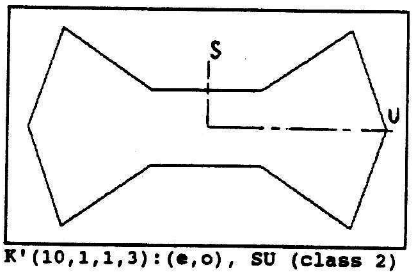
$$\mathbf{comp} = m, \tag{2.8}$$

suggesting (cf. Eq. 2.5) that we define [**sym**] = 1 for unbounded K'-patterns (patterns for which



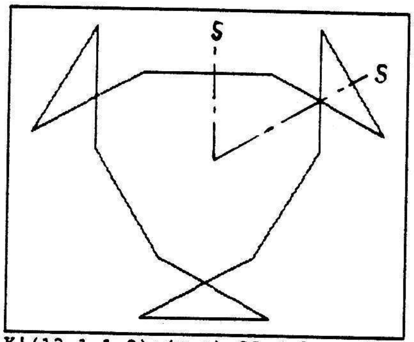
K'(80,6,1,3):(e,e), UU (class 1)

K'(80,6,1,3): (e,e), UU (class 1)
$\{d_4\}$



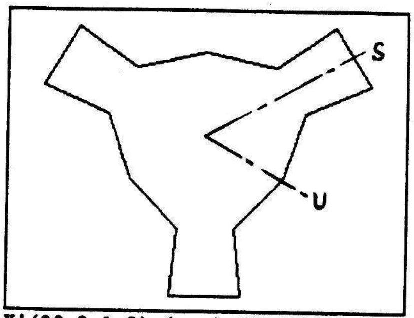
K'(10,1,1,3):(e,o), SU (class 2)

K'(10,1,1,3): (e,o), SU (class 2)
$\{d_2\}$



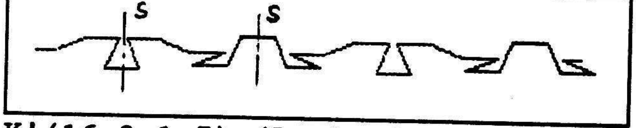
K'(12,1,1,3):(o,e) SS (class 3)

K'(12,1,1,3): (o,e), SS (class 3)
$\{d_3\}$



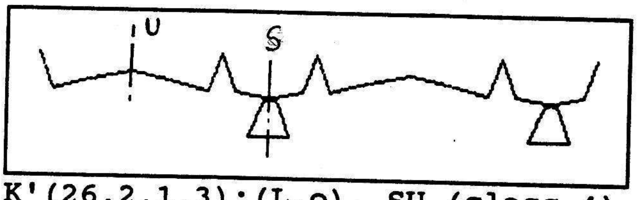
K'(30,2,1,3):(o,o) SU (class 4)

K'(30,2,1,3): (o,o), SU (class 4)
$\{d_3\}$



K'(16,3,1,5):(L,e), SS (class 3)

K'(16,3,1,5): ($\mathbb{L}$,e), SS (class 3)
$\{\mathbb{L}\}$



K'(26,2,1,3):(L,o), SU (class 4)

K'(26,2,1,3): ($\mathbb{L}$,o), SU (class 4)
$\{\mathbb{L})$

Fig. 2 Examples of K'-patterns belonging to the four parity classes

sym = $\mathbb{L}$). If we substitute *translation* for *rotation* in definition (b), we again obtain **comp** = m. (A frieze pattern may be regarded, after all, as a pattern with rotation symmetry of infinite order.)

Now that we have decided, on the basis of these heuristic arguments, to define [**sym**] = 1 for K'-patterns with $\mathbb{L}$ symmetry, we conclude that $\mathbb{L}$ patterns with even complexity are properly classified as (o,e) and therefore belong to parity class 3, while $\mathbb{L}$ patterns with odd complexity are properly classified as (o,o) and therefore belong to parity class 4. These assignments are also partly justified by the empirical observation that unbounded K-patterns of parity class (o,e) are invariably found to be of type SS, like their finite analogs. As in the case of bounded K'-patterns, no a priori reason has so far been discovered for ruling out the possibility of unbounded (o,e) K'-patterns of type UU. The examples of unbounded (o,o) K'-patterns, on the other hand, are *necessarily* of type SU, like their finite analogs (and for the same reason: the number of consecutive pattern vectors in a fundamental region of the reflection group of the pattern is half-integer).

3. Sufficient Condition for K'-patterns of Parity Class 2, 3, or 4 to Have Reflection Symmetry

From Fig. 3, it is evident that the existence of a solution to the congruence

$$(\lambda\text{-}js)^{\alpha} - \lambda^{\alpha} \equiv \lambda^{\alpha} - (\lambda+js)^{\alpha} \pmod{n}, \tag{3.1}$$

where

α = odd integer

$j = 0,1,2,\ldots,$**comp**/2

$\lambda = j_0 + Ys \quad (Y=0,1,2,\ldots,$**comp**/2)

is the necessary and sufficient condition for a K'-pattern of parity class 2, 3, or 4 to have reflection symmetry (since at least one configuration of type S must occur in every such pattern).

$$\text{angle } A = \frac{2\pi}{n}\left[(\lambda - js)^{\alpha} - \lambda^{\alpha}\right] \pmod{n}$$

$$\text{angle } B = \frac{2\pi}{n}\left[\lambda^{\alpha} - (\lambda + js)^{\alpha}\right] \pmod{n}$$

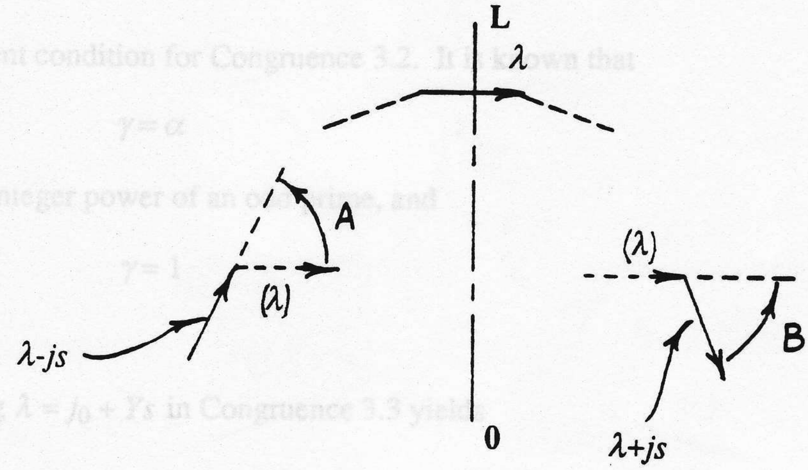


Fig. 3 Pattern Vectors for the Set of Arguments

$\ldots, \lambda - js, \lambda-(j-1)s, \ldots, \lambda-s, \lambda, \lambda+s, \ldots, \lambda+(j-1)s, \lambda+js, \ldots$

Type S Configuration

If a solution for Congruence 3.1 is sought in a linear search through the set of values $\lambda = j_0 + Ys$ ($Y = 0,1,2,\ldots,$**comp**$/2$), then the worst case time complexity is $O(n \log_2 \alpha)$. (Max(**comp**) = n, and using an optimal algorithm for computing α-power residues, the time is proportional to $\log_2 \alpha$.) A considerable simplification results if we cancel terms in Congruence 3.1, obtaining:

$$2\sum_{i=1}^{(\alpha-1)/2} \binom{\alpha}{2i} \lambda^{\alpha-2i} (js)^{2i} \equiv 0 \pmod{n} \tag{3.2}$$

(j=0,1,2,...,**comp**/2)

Now if we let $\gamma = \gcd\left\{\binom{\alpha}{2},\binom{\alpha}{4},\ldots,\binom{\alpha}{\alpha-1}\right\}$ we find that

$$2\gamma s^2 \lambda \equiv 0 \pmod{n} \tag{3.3}$$

is a sufficient condition for Congruence 3.2. It is known that

$$\gamma = \alpha$$

if α = an integer power of an odd prime, and

$$\gamma = 1$$

otherwise.

Substituting $\lambda = j_0 + Ys$ in Congruence 3.3 yields

$$2\gamma s^3\, Y \equiv -2\gamma s^2 j_0 \pmod{n} \tag{3.4}$$

A solution of Congruence 3.4 exists iff

$[\,2\gamma s^3,\, n\,] \mid (-2\gamma s^2 j_0)$ [7].

Finding a solution for λ requires the testing of Congruence 3.4 by means of a linear search through the set of values Y = {0,1,2,...,**comp**/2}, but the worst case time complexity is now only O(n), and with a smaller constant than in the O($n \log_2 \alpha$) case of Congruence 3.1.

For K'-patterns belonging to parity class 2, 3, or 4, if the parameters n, s, j_0, and α are chosen at random, it is found that Congruence 3.4 usually provides a solution to the problem of color symmetry. For the rare examples for which Congruence 3.4 has no solution[7], a solution of the original polynomial Congruence 3.1 can of course always be found by a linear search.

4. <u>Sufficient Condition for a K'-Pattern of Parity Class 1 to Have Reflection Symmetry</u>

$$\text{angle } A = \frac{2\pi}{n}\,[(\mu\text{-}s)^{\alpha} - (\mu\text{-}(j\text{+}1)s)^{\alpha}] \pmod{n}$$

$$\text{angle } B = \frac{2\pi}{n}\,[(\mu\text{+}js)^{\alpha} - \mu^{\alpha}] \pmod{n}$$

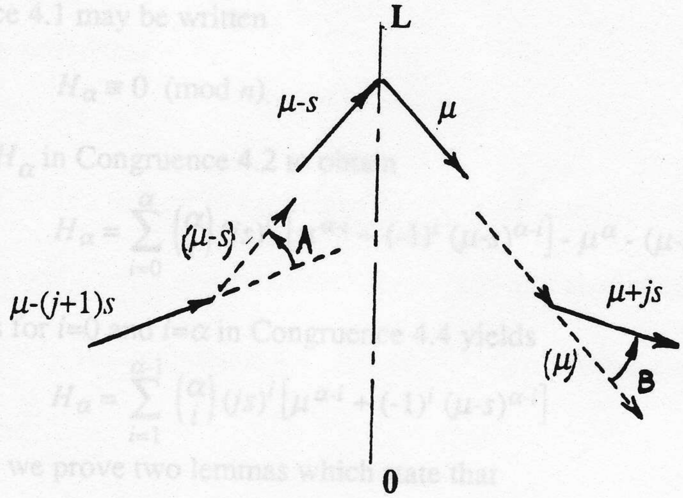


Fig. 4 Pattern Vectors for the Set of Arguments

$\ldots,\ \mu\text{-}js, \mu\text{-}(j\text{-}1)s, \ldots,\ \mu\text{-}s, \mu, \mu\text{+}s, \ldots,\ \mu\text{+}(j\text{-}1)s, \mu\text{+}js, \ldots$

Type U Configuration

From Fig. 4, it is evident that the necessary and sufficient condition for a K'-pattern of parity class 1 to be symmetrical by reflection is the existence of a solution of the congruence

$$(\mu\text{+}js)^{\alpha} - \mu^{\alpha} \equiv (\mu\text{-}s)^{\alpha} - [\mu\text{-}(j\text{+}1)s]^{\alpha} \pmod{n} \tag{4.1}$$

where

α = odd integer

$$j = 0,1,2,\ldots,\text{comp}/2$$

$$\mu = j_0 + Zs \quad (Z = 0,1,2,\ldots).$$

Of course it is not necessary here to restrict the K'-pattern to parity class 1; the analysis applies also to K'-patterns of classes 2 and 4, since at least one configuration of type U must occur in every such pattern. Let

$$H_\alpha = (\mu+js)^\alpha + [\mu-(j+1)s]^\alpha - [\mu^\alpha + (\mu-s)^\alpha]. \tag{4.2}$$

Then Congruence 4.1 may be written

$$H_\alpha \equiv 0 \pmod{n}. \tag{4.3}$$

We can expand H_α in Congruence 4.2 to obtain

$$H_\alpha = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (js)^i \left[\mu^{\alpha-i} + (-1)^i (\mu-s)^{\alpha-i}\right] - \mu^\alpha - (\mu-s)^\alpha \tag{4.4}$$

Cancelling terms for i=0 and $i=\alpha$ in Congruence 4.4 yields

$$H_\alpha = \sum_{i=1}^{\alpha-1} \binom{\alpha}{i} (js)^i \left[\mu^{\alpha-i} + (-1)^i (\mu-s)^{\alpha-i}\right] \tag{4.5}$$

In the Appendix, we prove two lemmas which state that

$$(2\mu-s) \mid [\mu^{\alpha-i} + (-1)^i (\mu-s)^{\alpha-i}] \quad (i=1,2,\ldots,\ \alpha-1). \tag{4.6}$$

Hence

$$(2\mu-s) \mid H_\alpha. \tag{4.7}$$

If we define

$$\gamma = \gcd\left\{\binom{\alpha}{i}\right\}_{i=1,2,\ldots,\ \alpha-1}$$

then

$$\gamma \mid H_\alpha \tag{4.8}$$

We will now prove that

$$2 \mid H_\alpha. \tag{4.9}$$

If either j or s is even -- i.e., if j,s = ee, eo, or oe -- Relation 4.9 follows immediately. For the case j,s = oo, Relation 4.9 follows directly from the identity

$$\binom{\alpha}{i} = \binom{\alpha}{\alpha-i} \quad (i=0,1,2,..., \ \alpha) \tag{4.10}$$

and from the fact that for all eight parity assignments (eee,eeo,eoe,eoo,oee, oeo, ooe, and ooo) to $\mu, s, i,$ the expressions

$$\mu^{\alpha-i} + (-1)^i (\mu-s)^{\alpha-i}$$

$$\mu^i + (-1)^{\alpha-i} (\mu-s)^i$$

have the same parity.

We will now prove that

$$s^2 \mid H_\alpha. \tag{4.11}$$

If i is even, then s^2 is a divisor of each term $(js)^i$ in Congruence 4.4, and therefore s^2 is a divisor of H_α. If i is odd, s is a divisor of each term $(js)^i$ and also of each factor $[\mu^{\alpha-i} + (-1)^i (\mu-s)^{\alpha-i}]$ in Congruence 4.4. Hence $s^2 \mid H_\alpha$. From Relations 4.7, 4.8, 4.9, and 4.11, it follows that

$$2\gamma s^2 (2\mu-s) \mid H_\alpha \tag{4.12}$$

and therefore that

$$2\gamma s^2 (2\mu-s) \equiv 0 \pmod n$$

$$\mu = j_0 + Zs \quad (Z = 0,1,2,...) \tag{4.13}$$

is a sufficient condition for Congruence 4.1.

If we substitute $\mu = j_0 + Zs$ in Congruence 4.13, we obtain

$$4\gamma s^3 Z \equiv 2\gamma s^2 (s-2j_0) \pmod n. \tag{4.14}$$

Now let $d = [4\gamma s^3, n]$. Then a solution of Congruence 4.14 exists iff

$$d \mid [2\gamma s^2 (s-2j_0)]. \tag{4.15}$$

For most of the large variety of K'-patterns of parity class 1 which have been examined, it is found that the divisibility relation 4.15 is satisfied. (When a solution is found for Z, additional solutions necessarily exist for $Z' = Z + i(2n/d)$ $(i=1,2,...,d-1)$.) For those cases for which the divisibility relation 4.15 is not satisfied, a solution of the polynomial congruence 4.1 may of course always be found by a linear search, just as in the case of parity classes 2, 3, or 4.

In an earlier analysis of this problem[3], it was proved that for K'-patterns belonging to parity classes 1, 2, or 4, if $\alpha = 3$ or $\alpha = 5$, a sufficient condition for the existence of a line of reflection symmetry (of unspecified configuration type -- i.e., *either* S or U) is

$$\gamma s^2 [2\mu + (\mathbf{comp} - 1)\, s] \equiv 0 \pmod{n} \tag{4.16}$$

$$\mu = j_0 + Zs \quad (Z=0,1,2,...).$$

Congruence 4.16 is derived from the polynomial congruence

$$[\mu+(j+1)s]^\alpha - [\mu+js]^\alpha + [\mu+Gs-(j+1)s]^\alpha - [\mu+Gs-js]^\alpha \equiv 0 \pmod{n} \tag{4.17}$$

where

$$G = \mathbf{comp} - 1. \tag{4.18}$$

This congruence expresses the equality of angles A and B in Fig. 5; R1' and R2' are half-lines of reflection of type U, and R is a half-line of reflection of type S if **comp** is odd and of type U if **comp** is even.

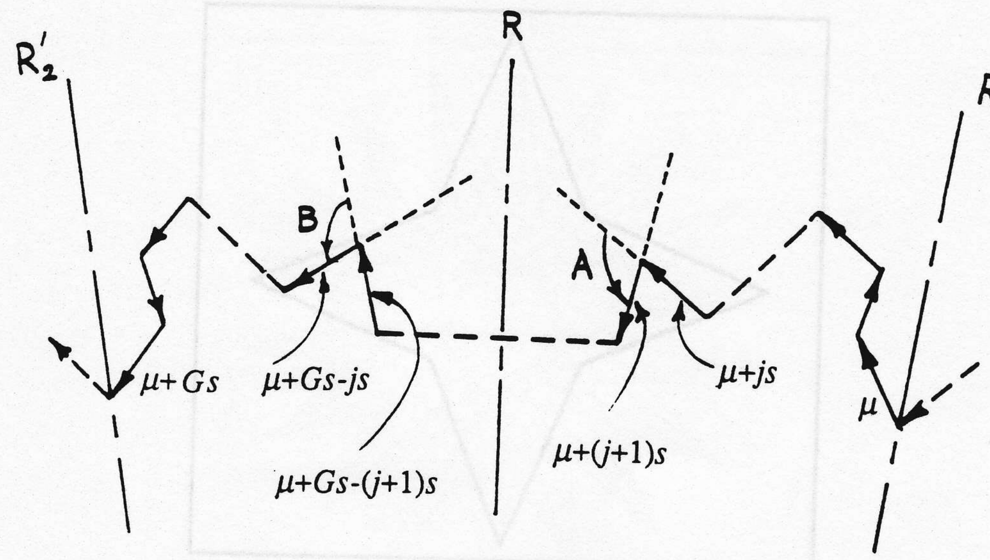


Fig. 5

An incomplete analysis of the cases $\alpha = 7$ and $\alpha = 9$ suggests that for K'-patterns belonging to parity classes 1, 2, or 4, Congruence 4.16 is quite possibly a sufficient condition for Congruence 4.17 to be satisfied for *all* odd α. A large variety of examples provide additional support for this conjecture.

A peculiarity of these symmetry conditions is indicated by the fact that there exist K'-patterns of parity class 1 for which a solution can be found from Congruence 4.16, but not from 4.14. A simple example is provided by the K'-pattern K'(16, 6, 1, 3, 8, 4), which is illustrated in Fig. 6.

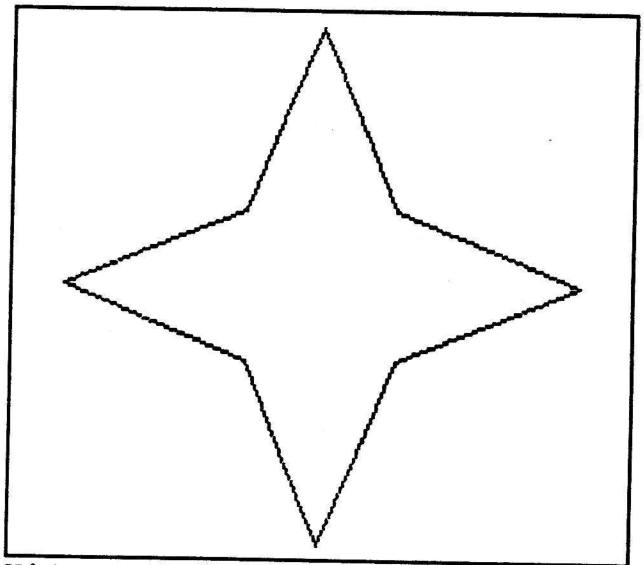
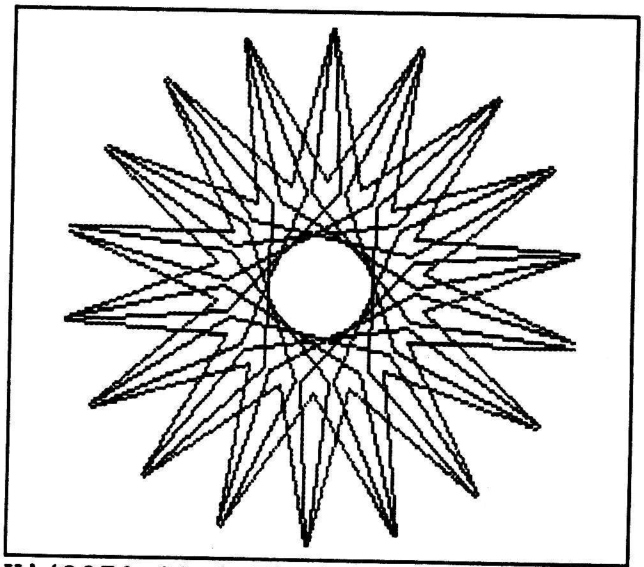


Fig. 6
The K'-pattern K'(16, 6, 1, 3, 8, 4) of parity class 1 for which Congruence 4.16 -- but not 4.14 -- provides a solution

For all of the examples so far encountered of K'-patterns which belong to parity class 2, and for which no solution is provided by Congruence 4.14, Congruence 4.16 *also* fails to provide a solution. The K'-pattern K'($2 \cdot 3^3 \cdot 5 \cdot 11$, $3 \cdot 11$, 1, 7, 90, 18), which is shown in Fig. 7, is an example of such a pattern. Until now, no examples have been found of K'-patterns of parity class 4 for which a solution fails to be provided by *both* Congruences 4.14 and 4.16.



K'(2970,33,1,7):(o,o)SU(class 4)

Fig.7

The K'-pattern K'($2\cdot3^3\cdot5\cdot11$, $3\cdot11$, 1, 7, 90, 9) of parity class 4 for which no solution is provided by either Congruence 4.14 or Congruence 4.16

NOTES

[1]For both bounded and unbounded patterns, m is the smallest integer which satisfies

$$[j_0 + s(j+m)]^{\alpha} \equiv (j_0 + sj)^{\alpha} \pmod{n} \tag{A.1}$$

for all integer values of j. For the special case $j_0 = 0$, the following theorem can be proved easily:

$$\text{Let } n = \prod_{i=1}^{r} p_i^{\nu_i} \tag{A.2}$$

and

$$s = s_0 \prod_{i=1}^{r} p_i^{\sigma_i} \quad ((s_0, p_i) = 1,\ i = 1,2,\ldots,\ r). \tag{A.3}$$

Then if $p_i \mid \alpha$ $(i = 1,2,\ldots,r)$,

$$m = \varepsilon \prod_{i=1}^{r} p_i^{\mu_i} \tag{A.4}$$

where μ_i

$$= \nu_i - \alpha\sigma_i \quad \text{if } \nu_i - \alpha\sigma_i \geq 0, \tag{A.5}$$

$$= 0 \quad \text{if } \nu_i - \alpha\sigma_i < 0, \tag{A.6}$$

and ε

$$= 2 \text{ if } n \text{ and } s \text{ are both odd}, \tag{A.8}$$

$$= 1 \text{ otherwise.}$$

[2]Proofs for the dependence of period and symmetry on n, s, j_0, α for a variety of families of K-patterns will be published elsewhere.

[3]K-Patterns, Technical Report, Dept. of Design, Southern Illinois University, Alan H. Schoen (Feb., 1984)

[4]The related problem of developing an optimal algorithm for locating a center of 2-fold rotational symmetry in K-patterns which have a single such rotocenter (or, for patterns with lattice symmetry, one such rotocenter per lattice fundamental region) has also been solved. An account of this work will be published elsewhere.

[5]It appears that it is *almost* correct to say that K-patterns with reflection symmetry occur only for odd α. Only one exception to this rule has been discovered thus far:

$$K(2,\ 1, 0, 2k, 2, \mathbb{L}^*) \quad (k = 1,2,3,...).$$

It is illustrated in Fig. A.1.

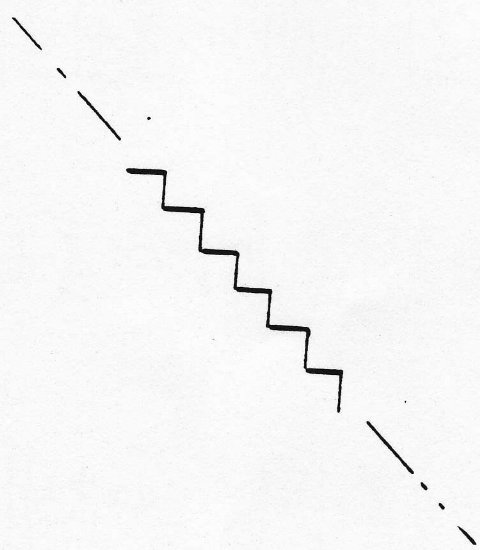


Fig. A.1

The symmetry group $\mathbb{L}^*$ of the pattern shown in Fig. A.1 includes both two [coaxial] half-turns and two [parallel] reflections; together these isometries define a glide reflection. We conjecture, without proof, that aside from the presumably small number of such exceptional degenerate cases, which are likely to occur only when the number of unit vectors in a single fundamental region of the translation (or rotation) group is small, *a necessary condition for a K-pattern to have reflection symmetry is that α be odd.* It is easy to exhibit counterexamples which prove that this is not also a *sufficient* condition.

[6]Introduction to Geometry, H. S. M. Coxeter, John Wiley & Sons, N. Y. (1961).

[7]The congruence $ax \equiv b \pmod{m}$ has solutions iff $d \mid b$; $d = [a,m]$. If $d \mid b$, there are exactly d solutions. If x_0 is a solution, then the other solutions are given by

$$x_0 + i(m/d) \quad (i = 1,2,\ldots,\ d-1).$$

APPENDIX

<u>Lemma 1:</u> $$\mu^{2n} - (\mu-2s)^{2n} = (2\mu-s)\mu^{2n-1}\sum_{i=1}^{2n-1}\left(\frac{s}{2\mu}\right)^{i}\sum_{j=1}^{i}(-2)^{j-1}\binom{2n}{j} \quad (n=1,2,3,\ldots). \tag{L1.1}$$

<u>Proof:</u> The l. h. s. of L1.1 can be written

$$\mu^{2n} - \sum_{i=0}^{2n}\binom{2n}{i}\mu^{2n-i}(-s)^{i} = -\sum_{i=1}^{2n-1}\binom{2n}{i}\mu^{2n-i}s^{i}(-1)^{i} - s^{2n}. \tag{L1.2}$$

If we set the r. h. s. of L1.2 equal to the r. h. s. of L1.1, we obtain

$$-s^{2n} = \sum_{i=1}^{2n-1}\binom{2n}{i}\mu^{2n-i}s^{i}(-1)^{i} + (2\mu-s)\,\mu^{2n-1}\sum_{i=1}^{2n-1}\left(\frac{s}{2\mu}\right)^{i}\sum_{j=1}^{i}(-2)^{j-1}\binom{2n}{j} \tag{L1.3}$$

Factoring yields

$$\left(\frac{s}{\mu}\right)^{2n+1} = \sum_{i=i}^{2n-1}\left[\left(\frac{s}{\mu}-2\right)\frac{1}{2^{i}}\sum_{j=1}^{i}(-2)^{j-1}\binom{2n}{j} + (-1)^{i+1}\binom{2n}{i}\right]\left(\frac{s}{\mu}\right)^{i} \tag{L1.4}$$

Let $x = s/\mu$. The L1.4 becomes

$$x^{2n} = (x-2)\left[\sum_{i=1}^{2n-1}\left(\frac{1}{2^{i}}\sum_{j=1}^{i}(-2)^{j-1}\binom{2n}{j}\right)x^{i}\right] + \left[\sum_{i=1}^{2n-1}(-1)^{i+1}\binom{2n}{i}x^{i}\right] \tag{L1.5}$$

or

$$x^{2n} = x^{2n}\left[\frac{1}{2^{2n-1}}\sum_{j=1}^{2n-1}(-2)^{j-1}\binom{2n}{j}\right] + \sum_{i=1}^{2n-2}\left[\frac{1}{2^{i}}\sum_{j=1}^{i}(-2)^{j-1}\binom{2n}{j}x^{i+1}\right]$$

$$-2\sum_{i=1}^{2n-1}\left[\left\{\frac{1}{2^{i}}\sum_{j=1}^{i}(-2)^{j-1}\binom{2n}{j}\right\}x^{i}\right] + \sum_{i=1}^{2n-1}\left[(-1)^{i+1}\binom{2n}{i}x^{i}\right]. \tag{L1.6}$$

To prove L1.6, we need only prove that

$$\text{(a)} \quad \sum_{j=1}^{2n-1} (-2)^{j-1} \binom{2n}{j} = 2^{2n-1} \qquad \text{(L1.*)}$$

and (b) that the coefficients of all powers of $x < 2n$ in L1.6 vanish identically.

L1.* is a simple consequence of the binomial expansion of $(2 - 1)^{2n} = 1$. The proof of (b) is easily accomplished after considering as separate cases the coefficient of x and the coefficient of x^i for $2 \leq i \leq 2n\text{-}1$. In the first case we find

$$\frac{-2}{2^i}(-2)^{i-1}\binom{2n}{i} + (-1)^{i+1}\binom{2n}{i} = \left[\frac{(-2)^i}{2^i} + (-1)^{i+1}\right]\binom{2n}{i}$$

$$= \left[(-1)^i + (-1)^{i+1}\right]\binom{2n}{i}$$

$$= 0.$$

In the second case, we obtain for the coefficient

$$\frac{1}{2^{i-1}}\sum_{j=1}^{i-1}(-2)^{j-1}\binom{2n}{j} - \frac{2}{2^i}\sum_{j=1}^{i}(-2)^{j-1}\binom{2n}{j} + (-1)^{i+1}\binom{2n}{i}$$

$$= \left(\frac{1}{2^{i-1}} - \frac{1}{2^{i-1}}\right)\sum_{j=1}^{i-1}(-2)^{j-1}\binom{2n}{j} + \left[-(-1)^{i-1} + (-1)^{i-1}\right]\binom{2n}{i}$$

$$= 0.$$

Hence L1.1 is proved.

Lemma 2:

$$\mu^{2n+1} + (\mu-s)^{2n+1} = (2\mu-s)\,\mu^{2n}\sum_{i=0}^{2n}\left(\frac{s}{2\mu}\right)^i\left[1+\frac{1}{2}\sum_{j=1}^{i}(-2)^j\binom{2n+1}{j}\right] \quad (n = 1,2,3,...). \qquad \text{(L2.1)}$$

Proof: The l. h. s. of L2.1 can be written

$$\mu^{2n+1} + \sum_{i=0}^{2n+1}\binom{2n+1}{i}\mu^{2n+1-i}(-s)^i = \mu^{2n+1} - s^{2n+1} + \mu^{2n+1}\sum_{i=0}^{2n}\binom{2n+1}{i}\mu^{-i}(-s)^i \qquad \text{(L2.2)}$$

If we set the r. h. s. of L2.2 equal to the r. h. s. of L2.1, we obtain

$$\mu^{2n+1} - s^{2n+1} = (2\mu-s)\,\mu^{2n}\sum_{i=0}^{2n}\left(\frac{s}{2\mu}\right)^i\left[1+\frac{1}{2}\sum_{j=1}^{i}(-2)^j\binom{2n+1}{j}\right] - \sum_{i=0}^{2n}\binom{2n+1}{i}\left(\frac{-s}{\mu}\right)^i \qquad \text{(L2.3)}$$

Factoring yields

$$1 - \left(\frac{s}{\mu}\right)^{2n+1} = \frac{1}{2}\left(2-\frac{s}{\mu}\right)\sum_{i=0}^{2n}\left(\frac{s}{2\mu}\right)^i\left[1+\sum_{j=0}^{i}(-2)^j\binom{2n+1}{j}\right] - \sum_{i=0}^{2n}\binom{2n+1}{i}\left(\frac{-s}{\mu}\right)^i \qquad \text{(L2.4)}$$

Let $x = s/\mu$. Then L2.4 becomes

$$x^{2n+1} = (x-2)\sum_{i=0}^{2n}\frac{1}{2^{i+1}}\left[1+\sum_{j=0}^{i}(-2)^j\binom{2n+1}{j}\right]x^i + \sum_{i=0}^{2n}(-1)^i\binom{2n+1}{i}x^i + 1 \qquad \text{(L2.5)}$$

or

$$x^{2n+1} = x^{2n+1}\left[\frac{1}{2^{2n+1}}\left\{1+\sum_{j=0}^{2n}(-2)^j\binom{2n+1}{j}\right\}\right] + \sum_{i=0}^{2n-1}\frac{1}{2^{i+1}}\left[1+\sum_{j=0}^{i}(-2)^j\binom{2n+1}{j}\right]x^{i+1}$$

$$- 2\sum_{i=0}^{2n}\frac{1}{2^{i+1}}\left[1+\sum_{j=0}^{i}(-2)^j\binom{2n+1}{j}\right]x^i + \sum_{i=0}^{2n}(-1)^i\binom{2n+1}{i}x^i + 1. \qquad \text{(L2.6)}$$

To prove L2.6, we need only prove that

$$\text{(a)}\quad 1+\sum_{j=0}^{2n}(-2)^j\binom{2n+1}{j} = 2^{2n+1} \qquad \text{(L2.*)}$$

and (b) that the coefficients of all powers of $x < 2n+1$ in L2.6 vanish identically. L2.* is a simple consequence of the binomial expansion of $(-2+1)^{2n+1} = -1$. The proof of (b) is easily accomplished if we consider as separate cases the coefficient of x^0 (i.e., the case $i = 0$) and the coefficient of x^i for $1 \le i \le 2n$. In the first case, we find

$$-2\cdot\frac{1}{2}\left[1+(-2)^0\binom{2n+1}{0}\right]x^0 + (-1)^0\binom{2n+1}{0}x^0 + 1$$

$$= -[1+1]+1+1$$

$$= 0.$$

In the second case, we obtain for the coefficient

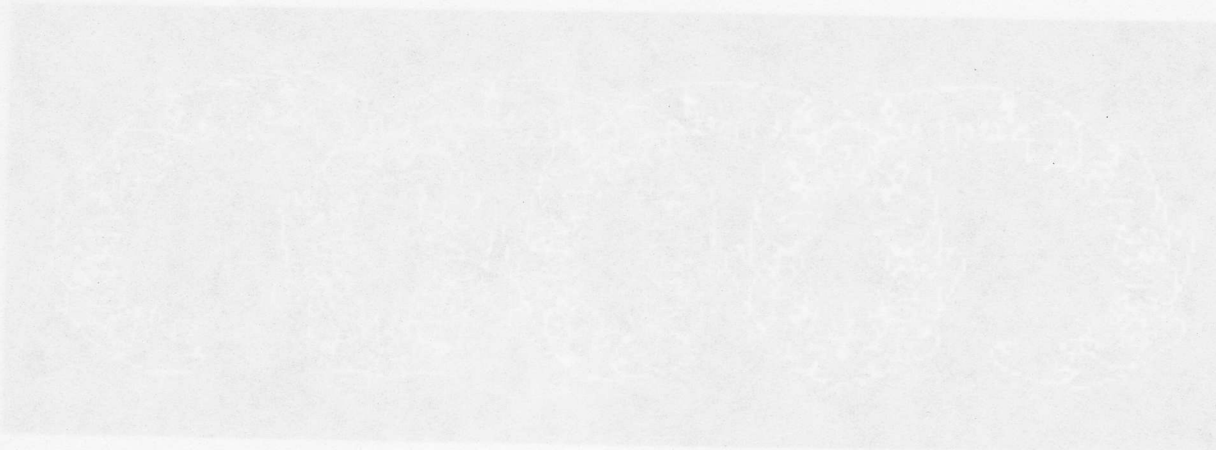
$$\frac{1}{2^i}\left[1+\sum_{j=0}^{i-1}(-2)^j\binom{2n+1}{j}\right] - \frac{1}{2^i}\left[1+\sum_{j=0}^{i}(-2)^j\binom{2n+1}{j}\right] + (-1)^i\binom{2n+1}{i}$$

$$\frac{1}{2^i}\left[\sum_{j=0}^{i-1}(-2)^j\binom{2n+1}{j}\right] - \frac{1}{2^i}\left[\sum_{j=0}^{i-1}(-2)^j\binom{2n+1}{j} + (-2)^i\binom{2n+1}{i}\right] + (-1)^i\binom{2n+1}{i}$$

$$= \left[-(-1)^i + (-1)^i\right]\binom{2n+1}{i}$$

$$= 0.$$

Hence L2.1 is proved.



SOME EXAMPLES OF K-PATTERNS