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A CONSTRUCTION METHOD FOR TRIPLY PERIODIC MINIMAL SURFACES

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Abstract. A uniform and elementary treatment of many classical and new embedded triply periodic minimal surfaces in Euclidean space, based on a Schwarz-Christoffel formula for periodic polygons in the plane, is given.

Introduction

A triply periodic minimal surface is a minimally immersed surface in Euclidean 3-space $\mathbb{R}^3$ which is invariant under translations in three independent directions (the three translations generate a lattice Λ in $\mathbb{R}^3$).

Throughout this article, we assume triply periodic minimal surfaces to be non-flat and properly embedded.

In the nineteenth century, H. A. Schwarz and his students found five triply periodic minimal surfaces [Schw, Neo]. See Figure 1.

In 1970, A. Schoen described twelve more examples [Scho]. The pictures of his examples can be found in the web pages [Br, We].

In 1989, H. Karcher proved the existence of Schoen's surface and found more examples, using the conjugate surface method [Ka1]. See also [Ka2, KaPo].

Since then, a vast numbers of examples have been found by many people [FiKo1, FiKo2, Hu, Me, Ram, Tr1, Tr2].

In a recent work, the authors gave a construction method for embedded triply periodic minimal surfaces [FuWe]. These surfaces share the property that vertical symmetry planes cut them into simply connected pieces. The Weierstrass representation of these surfaces can be given in terms of a Schwarz-Christoffel formula for *periodic* Euclidean polygons, using ϑ-functions on suitable tori.

In this article, we briefly introduce this construction method and give examples of embedded triply periodic minimal surfaces. We do not give any proofs here, so for details, see [FuWe].

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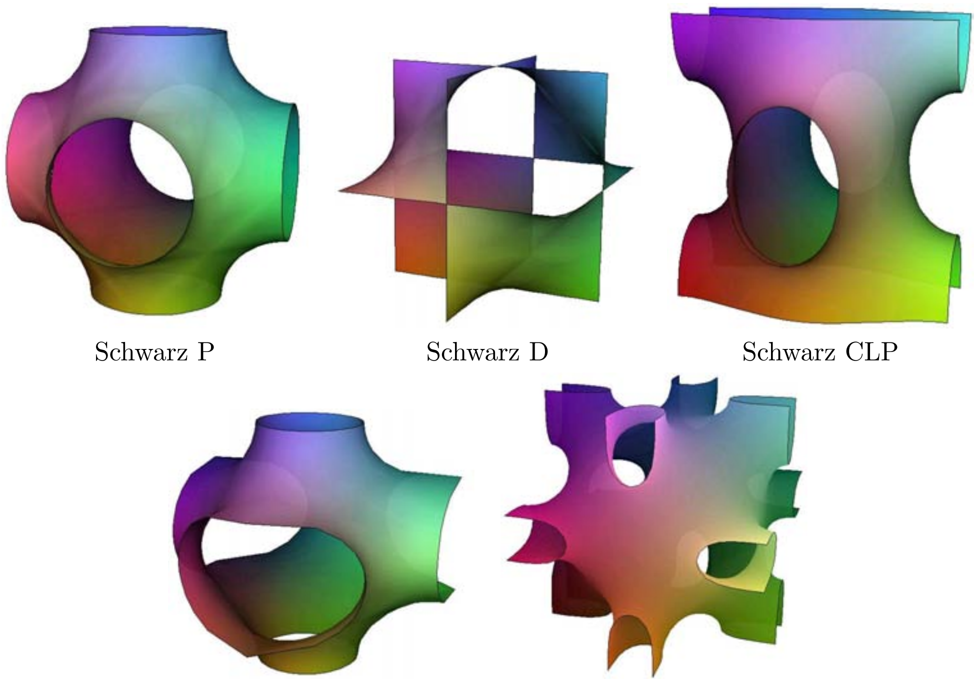


FIGURE 1. Classical examples of triply periodic minimal surfaces.

A triply periodic minimal surface can be considered as a minimal embedding from a compact Riemann surface of genus $g \geq 3$ into a flat 3-torus $\mathbb{R}^3/\Lambda$.

Table 1 shows the genus of the triply periodic minimal surfaces known to Schwarz and Schoen.

Name	Genus	Name	Genus	Name	Genus
Schwarz P	3	Schwarz D	3	Schwarz CLP	3
Schwarz H (Schoen RI)	3	Neovius (Schoen C(P))	9		
Gyroid	3	Schoen H'-T	4	Schoen S'-S"	4
Schoen I-WP	4	Schoen H"-R	5	Schoen T'-R'	6
Schoen F-RD	6	Schoen C(H)	7	Schoen RII	9
Schoen O,C-TO	10	Schoen RIII	13	Schoen C(D)	19

TABLE 1. Genus of Schwarz and Schoen's surfaces.

Our construction method consists of the following 2 steps:

Step 1 We construct a minimal embedding defined on a parallel strip with the following properties:

- it is bounded by a vertical prism over a triangle with angles π/r, π/s and π/t, where $(r,s,t) \in \{(2,3,6),\ (2,4,4),\ (3,3,3)\}$,
- the boundary curves lie on the prism,
- it intersects the prism orthogonally,
- it is invariant under a vertical translation.

Step 2 We repeat reflection with respect to a vertical plane of the prism suitably many times.

To construct a surface satisfying the above properties, we use an equivariant Schwarz-Christoffel formula.

In the next section we will discuss about this.

1. An equivariant Schwarz-Christoffel formula

In this section, we will introduce an equivariant version of the classical Schwarz-Christoffel formula for periodic polygons.

The classical Schwarz-Christoffel mapping is a biholomorphic mapping from the upper half-plane into a polygon. It can be explicitly written down the following formula:

$$f(z) = \int^z \prod_{i=1}^m (z-p_i)^{a_i}\,dz,$$

where $p_i \in \mathbb{R} \cup \{\infty\}$ and $f(p_i)$ correspond to the vertices of the polygon with angles $(a_i+1)\pi$. See [Neh] for details.

Instead of the monomial factors $z - p_i$ of the classical Schwarz-Christoffel formula, we will use ϑ-factors $\vartheta(z-p_i)$, where

$$\vartheta(z) = \vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+\frac{1}{2})^2\tau + 2\pi i (n+\frac{1}{2})(z-\frac{1}{2})}$$

is one of the classical Jacobi ϑ-functions [Mu]. It is an entire function with simple zeroes at the lattice points of the integer lattice spanned by 1 and τ. It satisfies the following symmetries:

$$\vartheta(-z) = -\vartheta(z), \quad \vartheta(z+1) = -\vartheta(z), \quad \vartheta(z+\tau) = -e^{-\pi i\tau - 2\pi i z}\vartheta(z).$$

Furthermore, $\vartheta(\bar{z}) = \overline{\vartheta(z)}$ if $\tau \in i\mathbb{R}$.

Definition 1. A *periodic polygon* P is a simply connected domain in the plane $\mathbb{C}$ with the following properties:

- P is bounded by two infinite-length piecewise linear curves with discrete vertex sets,
- P is invariant under a Euclidean translation $V(z) = z + v$ for some $v \in \mathbb{C} - \{0\}$,
- The quotient $P/\langle V \rangle$ is conformally an annulus.

See Figure 2.

We denote the vertices of the two boundary curves by P_i and Q_j so that $V(P_i) = P_{i+m}$ and $V(Q_j) = Q_{j+n}$ for all i,j and fixed integers m,n. Denote the interior angles of the polygon at P_i (resp. Q_j) by α_i (resp. β_j). By assumption, these numbers are also periodic with respect to m and n, respectively.

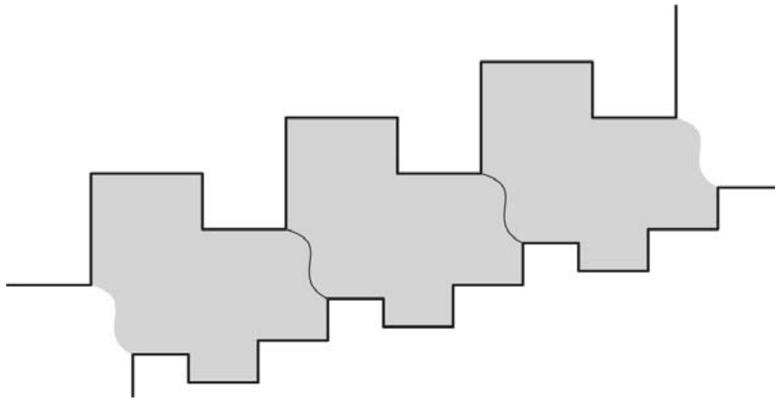


FIGURE 2. A periodic polygon

Because both boundary curves are invariant under a translation, we see that

$$\sum_{i=1}^{m}(\pi-\alpha_i)=0=\sum_{j=1}^{n}(\pi-\beta_j). \tag{1}$$

Let P be a periodic polygon invariant under V. Let $d/2$ be the modulus of the annulus $P/\langle V\rangle$, and define the strip $Z=\{z\in\mathbb{C}:0<\operatorname{Im}z<d/2\}$. The choice of d makes the annuli $Z/\langle h\rangle$ and $P/\langle V\rangle$ conformally equivalent, and $d>0$ is uniquely determined this way, where $h:Z\to Z$ is defined by $h(z)=z+1$. Moreover, we obtain a biholomorphic map $f:Z\to P$ which is equivariant with respect to both translations, that is, the following diagram is commutative:

$$\begin{array}{ccc} Z & \xrightarrow{f} & P \\ h\big\downarrow & & \big\downarrow V \\ Z & \xrightarrow{f} & P \end{array}$$

This map extends to a homeomorphism between the closures of Z and P. In comparison to the classical Schwarz-Christoffel formula, Z will play the role of the upper half plane.

Proposition 2 ([FuWe, Proposition 2.3])**.** *Let P be a periodic polygon, and Z be the associated parallel strip as above. Then, up to scaling, rotating, and translating,*

$$f(z)=\int^{z}\prod_{i=1}^{m}\vartheta(z-p_i)^{a_i}\cdot\prod_{j=1}^{n}\vartheta(z-q_j)^{b_j}\,dz$$

is the biholomorphic map from Z to P. Conversely, for any choices of $p_i\in\mathbb{R}$ $(1\le i\le m)$, $q_j\in\mathbb{R}+di/2$ $(1\le j\le n)$ and $-1<a_i,b_j<1$ satisfying the angle condition

$$\sum_{i=1}^{m}a_i=0=\sum_{j=1}^{n}b_j, \tag{2}$$

f maps Z to a periodic polygon.

2. Minimal surfaces defined on parallel strips

In this section we construct a minimal surface which satisfies the properties of Step 1 as in the introduction. The Weierstrass data can be given by the integrands of our Schwarz-Christoffel formula.

Any conformally parametrized minimal surface in $\mathbb{R}^3$ can locally be given by the Weierstrass representation

$$z \mapsto \operatorname{Re} \int^z \left(\frac{1}{2}\left(\frac{1}{G} - G\right), \frac{i}{2}\left(\frac{1}{G} + G\right), 1 \right) dh,$$

where G is a meromorphic function and dh is a holomorphic 1-form. G can be identified with the Gauss map of the surface via the stereographic projection, and dh is called the height differential of the surface. The pair (G, dh) is called the Weierstrass data of the minimal surface.

For a positive real number d, let $Z = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < d/2\}$ be a strip domain.

For $m, n, \geq 0$ consider points $p_i \in (0,1), i = 1, \dots, m$ and $q_j \in (0,1) + \tau/2, j = 1, \dots, n$. Extend these to periodic sets of points by imposing the condition $p_{i+m} = p_i + 1$, $q_{j+n} = q_j + 1$ for all i, j.

Consider the minimal surface defined on Z by the Weierstrass data

$$G(z) = \prod_{i=1}^{m} \vartheta(z - p_i)^{a_i} \cdot \prod_{j=1}^{n} \vartheta(z - q_j)^{b_j}, \qquad \text{and} \qquad dh = dz.$$

For the exponents we assume that $-1 < a_i, b_j < 1$ for all i, j (this ensures that all interior angles of the Schwarz-Christoffel polygons are between 0 and 2π). We also assume the angle condition (2). Then we have:

Proposition 3 ([FuWe, Proposition 3.1])**.** *The minimal surface with the above Weierstrass data is a simply connected minimal surface with two boundary components lying in a finite number of vertical symmetry planes. These planes meet at angles πa_i at the image of p_i and πb_j at the image of q_j. Furthermore, the surface is invariant under the vertical translation $x_3 \mapsto x_3 + 1$.*

Definition 4. For an interval $[p_i, p_{i+1}]$ or $[q_j, q_{j+1}]$, we denote the vertical symmetry plane in which the corresponding planar symmetry curve lies by $\Pi_{[p_i, p_{i+1}]}$ or $\Pi_{[q_j, q_{j+1}]}$. We orient these planes by insisting that their normal vector points away from the minimal surface.

3. The Basic Examples

The simplest example of a periodic polygon has just two vertices along one edge and none along the other, that is, $m = 2$ and $n = 0$. We will now discuss the resulting minimal surface. By translating Z, we can assume that $p_1 = -p$ and $p_2 = +p$. Furthermore, by the angle condition (2), $a_1 = a = -a_2$. Thus the Gauss map is given by

$$G(z) = \left(\frac{\vartheta(z - p)}{\vartheta(z + p)} \right)^a$$

for some $0 < a < 1$.

With this normalization, the minimal surface becomes symmetric with respect to a reflection at the imaginary axis in the domain which corresponds to a reflection at a horizontal plane in space, which we can assume to be the x_1x_2-plane.

By Proposition 3, the minimal surface has two boundary arcs. The one corresponding to $\operatorname{Im} z = \operatorname{Im} \tau/2$ lies in a single coordinate plane Π, while the other switches between coordinate planes $\Pi_{[-p,p]}$ and $\Pi_{[p,-p]}$, making an angle $a\pi$. The angle between Π and $\Pi_{[-p,p]}$ can be computed by the following formula:

Proposition 5 ([FuWe, Proposition 4.1])**.** *The angle* α_0 *between* $\Pi_{[-p,+p]}$ *and* Π *is equal to*

$$\alpha_0 = \pi a(2p-1).$$

Our goal is to obtain a triply periodic surface from this minimal surface by repeated reflections at the three symmetry planes. In order for the surface to be embedded, the group generated by these reflection must be a Euclidean triangle group.

There are three such groups, denoted by $\Delta(2,3,6)$, $\Delta(2,4,4)$, and $\Delta(3,3,3)$, where $\Delta(r,s,t)$ corresponds to the group generated by reflecting at the edges of a triangle with angles π/r, π/s, and π/t. We assign a specific such triangle group to Σ as follows:

The angle at $+p$ is to be π/r, the angle between $\Pi_{[-p,p]}$ and Π equals π/s, and the angle between $\Pi_{[p,-p]}$ and Π shall be π/t. By Proposition 3, the first condition forces

$$a = \frac{r-1}{r},$$

while the second determines

$$p = \frac{-r-s+rs}{2(r-1)s}$$

by Proposition 5.

This allows for 10 possibilities. However, due to the reflectional symmetry at the x_1x_2-plane, any surface corresponding to $\Delta(r,s,t)$ becomes one corresponding to $\Delta(r,t,s)$, by turning it upside down. Thus there are only 6 distinct cases. They all correspond to surfaces known to Schwarz [Schw] or Schoen [Scho]. See Figure 3.

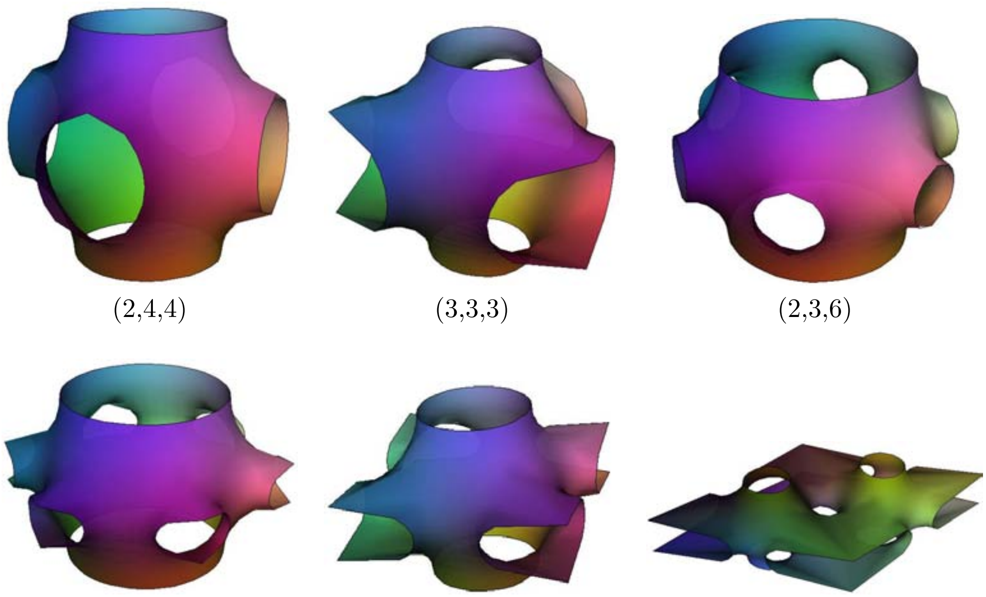
The following table lists all cases with the naming convention of Schoen, as well as the value of p and their genus.

Name	(r,s,t)	p	genus
Schwarz P	(2,4,4)	1/4	3
Schoen H'-T	(2,6,3)	1/3	4
Schoen H'-T	(2,3,6)	1/6	4
Schwarz H	(3,3,3)	1/4	3
Schoen H"-R	(3,2,6)	1/8	5
Schoen H"-R	(3,6,2)	3/8	5
Schoen S'-S"	(4,4,2)	1/3	4
Schoen S'-S"	(4,2,4)	1/6	4
Schoen T'-R	(6,2,3)	1/5	6
Schoen T'-R	(6,3,2)	3/10	6

All these surfaces come in a 1-parameter family where $\tau \in i\mathbb{R}^+$ is the parameter.

Remark 6. To compute the genus of surfaces, we first observe that

$$G(z+1) = G(z) \qquad \text{and} \qquad G(z+\tau) = e^{4\pi i a p} G(z).$$



(2,4,4) (3,3,3) (2,3,6)

(3,2,6) (4,2,4) (6,2,3)

FIGURE 3. The basic examples.

Thus, for instance, in the (2,4,4)-surface case, we see that

$$G(z+1)^2 = G(z)^2 \qquad \text{and} \qquad G(z+2\tau)^2 = G(z)^2$$

and hence G is well-defined on the double covering of the torus $\mathbb{C}/\langle 1, 2\tau\rangle$ with four branch points at $\pm p$ and $\pm p + \tau$. Since each branch point has the branch order one, the Riemann-Hurwitz formula yields that the Riemann surface in which the Weierstrass data are well-defined on is of genus 3.

4. The next least complicated case

The next least complicated case after the basic case with two corners allows for four corners. We assume that the surface is symmetric with respect to the x_1x_2-plane. We can assume that this reflection is realized in the domain by a reflection at the imaginary axis.

Four corners can either lie in a single boundary component, or be divided into two corners for each component. However, the only possibility of the first case is an (unbranched) double cover over our basic examples by doubling the vertical period. See Section 6 of [FuWe].

So in this section we will discuss the second case, that is, we will discuss symmetric polygons where each boundary component has just two corners, that is, $m = n = 2$.

Without loss of generality we can assume that $p_1 = p = -p_2 \in (0, 1/2)$ and $q_1 = q = -q_2 \in (0, 1/2) + \tau/2$. We denote $a_1 = a = -a_2$ and $b_1 = b = -b_2$. Then the Gauss map is given by

$$G(z) = \left(\frac{\vartheta(z-p)}{\vartheta(z+p)}\right)^a \left(\frac{\vartheta(z-q)}{\vartheta(z+\bar{q})}\right)^b.$$

There are two qualitatively distinct cases, depending on whether a and b have the same or opposite signs.

4.1. **Exponents have equal signs.** Without loss of generality, we can assume that $a, b < 0$.

In this case, projecting the two boundary arcs into the x_1x_2-plane gives two "hinges" with angles πa and πb. We denote the hinge containing the image of p by H_p, and the hinge containing q by H_q.

We want these hinges to be part of our reflection group triangles. This requires a and b to be of the form $a = -(r-1)/r$ and $b = -(s-1)/s$ with $r, s \in \{2, 3, 4, 6\}$.

As the hinges have four edges all together, two of these edges must lie on one side of a reflection group triangle. By relabeling the vertices, if necessary, we can thus assume that the planes $\Pi_{[-p,p]}$ and $\Pi_{[-q,q]}$ are parallel. We have the following:

Proposition 7 ([FuWe, Proposition 5.1])**.**

$$a(2p-1) + b(2\,\mathrm{Re}(q) - 1) = 1. \tag{3}$$

This constraint between p and q guarantees that the planes $\Pi_{[-p,p]}$ and $\Pi_{[-q,q]}$ are parallel, but we need them to be equal. The following theorem shows that we can always adjust p to make this happen:

Theorem 8 ([FuWe, Theorem 7.1])**.** *For any $\tau \in i\mathbb{R}^+$ and $-1 < a, b < 0$, there exist $0 < p, \mathrm{Re}(q) < 1/2$ so that the two hinges H_p and H_q line up as a triangle.*

Idea of the proof. We fix $\tau \in i\mathbb{R}^+$ and $-1 < a, b < 0$. Then by (3), we have one free parameter p. To show the theorem, we have to adjust p so that the period problem is solved. We define a function Φ on Z as follows:

$$\Phi(z) = \int^z G dh.$$

Then the period condition is equivalent to that the image edges $\Phi([-p,p])$ and $\Phi([-q,q])$ are to be collinear.

Since the limit situations ($p \to 0$ and $q \to \tau/2$) can be analyzed explicitly, an intermediate value argument yields the existence of p so that the period problem is solved. □

The following table lists all possibilities of admissible angle combinations, except the cases of equal angles (that is, $r = s = 3$ or $r = s = 4$). These cases reduce to the $(r, 2, t)$ cases of Section 3, because the two hinges become symmetric, each being part of a $(r, 2, t)$-triangle.

r	s	Relation between p and q	genus
2	3	$6p + 8\,\mathrm{Re}(q) = 1$	8
2	4	$4p + 6\,\mathrm{Re}(q) = 1$	6
3	6	$8p + 10\,\mathrm{Re}(q) = 3$	10
6	2	$5p + 3\,\mathrm{Re}(q) = 1$	9

There is a more intuitive way to understand the resulting surfaces geometrically, namely as *superpositioning* two of the basic cases with the same underlying triangle group on top of each other. For instance, the $(2,4,4)$-surface (aka Schwarz P) and the $(4,4,2)$-surface (aka Schoen S'-S") from the basic case family can be combined into the $(2,4,4)$-surface of the current family. See Figure 4.

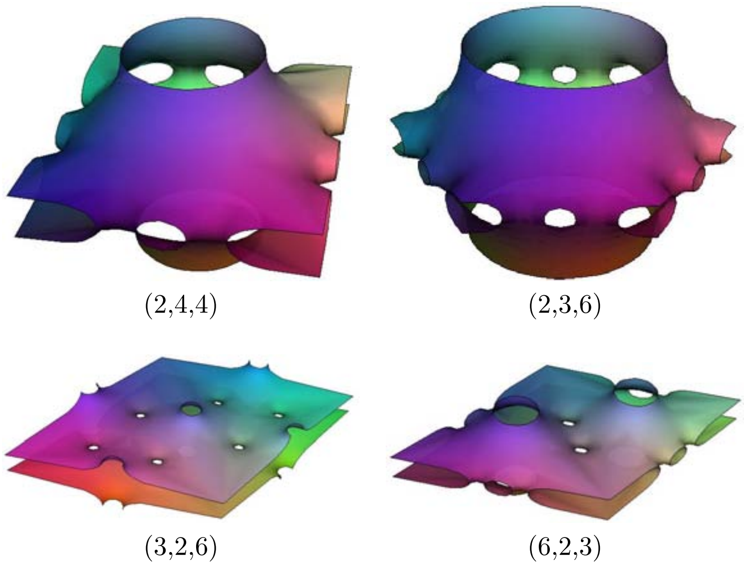


FIGURE 4. Four corners case: equal exponents.

4.2. **Exponents have opposite signs.** Without loss of generality, we can assume that $a < 0$ and $b > 0$.

In this case, the two hinges arrange themselves above each other and need to be matched up so that two edges are parallel and of the same length.

We normalize the divisors such that the edges $[-p, p]$ and $[q, 1-q]$ are to be matched. Let $a = -(r-1)/r$ and $b = (s-1)/s$.

This implies that we need the angle α_0 between $\Pi_{[-p,p]}$ and $\Pi_{[-q,q]}$ to be equal to π/s.

We have the following constraint:

Proposition 9 ([FuWe, Proposition 5.1])**.**

$$a(2p-1) + b(2\operatorname{Re}(q) - 1) = \frac{1}{s}.$$

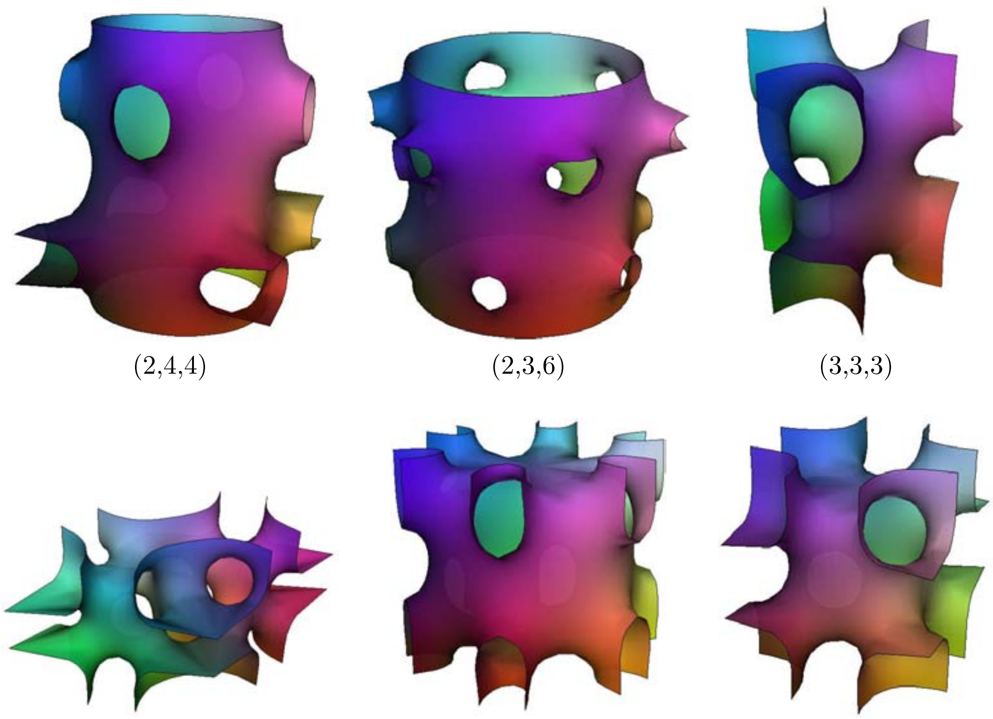
Again, all combinatorially possible cases are listed in the table below. Two of the cases have been discovered earlier by Schoen (I-WP) and Karcher (T-WP). These are particularly simple in that $r = s$. This allows for more symmetric solutions to the period problem with $-p + \operatorname{Re}(q) = 1/(2(r-1))$. This results in horizontal straight lines on the surfaces, namely as images of the vertical lines through p, q and $-p, -q$. Thus, the period problem is solved automatically. For these two cases, our construction method gives the double cover of the surfaces. Thus if we compute the genus of $(r, s) = (4, 4)$

case (resp. $(r,s) = (3,3)$ case), we see that it has genus 7 (resp. genus 5), but the actual I-WP surface (resp. T-WP surface) has genus 4 (resp. genus 3).

The other four surfaces are probably new.

Name	r	s	Relation between p and q	g
	2	3	$6p - 8\,\mathrm{Re}(q) = -3$	8
	2	4	$2p - 3\,\mathrm{Re}(q) = -1$	6
	3	6	$4p - 5\,\mathrm{Re}(q) = -1$	10
	6	2	$10p - 6\,\mathrm{Re}(q) = -1$	9
Schoen I-WP	4	4	$6p = 3\,\mathrm{Re}(q) = 1$	7 (4)
Karcher T-WP	3	3	$24p = 8\,\mathrm{Re}(q) = 3$	5 (3)

There is also a more intuitive way to understand the resulting surfaces geometrically, namely as *vertically stacking* two of the basic cases with the same underlying triangle group on top of each other. For instance, the $(2,4,4)$-surface (aka Schwarz P) and the $(4,4,2)$-surface (aka Schoen S'-S") from the basic case family can be combined into the $(2,4,4)$-surface of the current family. See Figure 5.



(2,4,4) (2,3,6) (3,3,3)

(3,2,6) (6,2,3) (4,2,4)

FIGURE 5. Four corners case: opposite exponents.

Note that any solution pair (p,q) to the period problem is isometric to $(1/2-p, 1/2-q)$ by shifting the divisor by $1/2$ and taking the reciprocal. The constraint equation will be slightly different then.

The following period problem can be solved using an extremal length argument.

Theorem 10 ([FuWe, Theorem 7.2])**.** *For any $-1 < a < 0$ and $0 < b < 1$, there is a 1-parameter family of values of any $\tau \in i\mathbb{R}^+$ such that the union of the hinges H_p and H_q forms a triangle.*

5. Higher Genus

Our method can be used to create many further examples of triply periodic minimal surfaces, by adding corners to the periodic polygons. This, however, also increases the dimension of the period problem. While there are methods available to solve such problems, it does not appear to be worth the effort at this point.

We briefly discuss one more case which give examples of higher genus surfaces, which are related to surfaces that have been discussed in the literature.

We again assume the reflectional family about the x_1x_2-plane. The lower edge is to have two corners at $\pm p$, while we assume four corners in the upper edge at $\pm q_1 + \tau/2$ and $\pm q_2 + \tau/2$. We label the exponents at p, q_1, q_2 as a, b_1, b_2.

We define the exponents by

$$a = \frac{1}{r} - 1 < 0, \qquad b_1 = 1 - \frac{1}{s} > 0, \qquad b_2 = 1 - \frac{1}{t} > 0.$$

In the case $(r, s, t) = (2, 4, 4)$, we obtain Neovius surface (in the case of full cubic symmetry, [Neo]). Also, in the case $(r, s, t) = (3, 3, 3)$, we obtain Schoen C(H) surface [Scho]. See also [Hu]. These two cases allow for an additional symmetry with $p = 1/4$ and $q_1 + q_2 = 1/2$, which renders the period problem 1-dimensional.

All other cases require to solve a 2-dimensional period problem, and lead to 1-parameter families, whose existence we have established numerically. See Figure 6.

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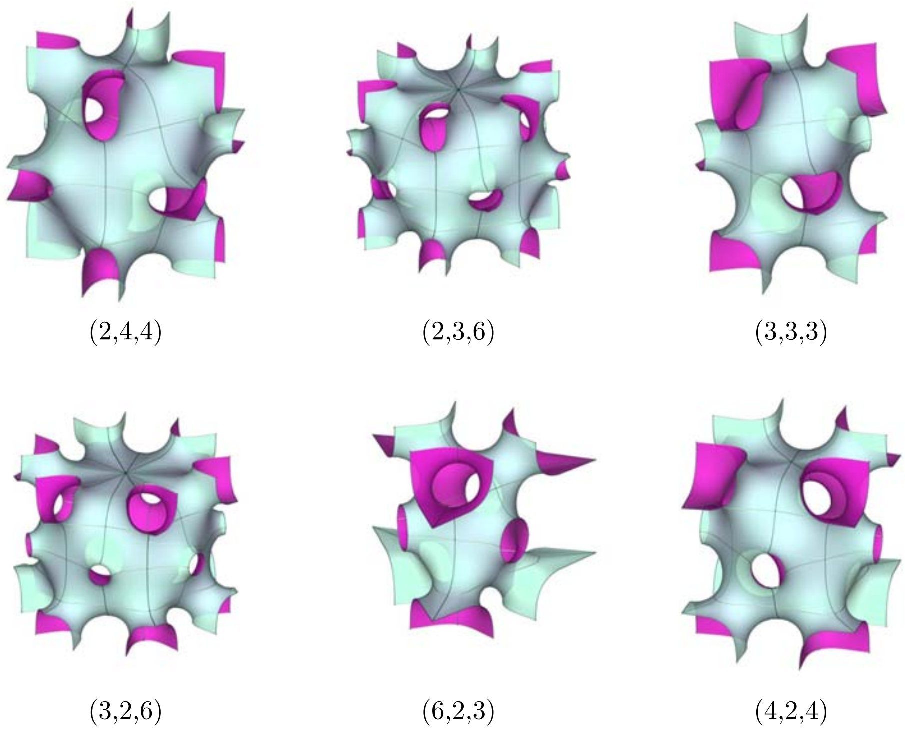

FIGURE 6. Neovius family.

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