Expansion of pointed mirror edge length in each generation

Theorem 1:

Let

$$\boldsymbol{\sigma}_k = (\sigma_k(1), \sigma_k(2), \sigma_k(3)), \text{ where } \sigma_k(i) \in \mathbf{Z}\, (i\text{=1,2,3}; \; k = 1,2,\ldots) \tag{1}$$

$$\boldsymbol{E}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{2}$$

$$\boldsymbol{\sigma}_{k+1} = \boldsymbol{E}_3 \boldsymbol{\tau}_k^{\mathsf{T}} \tag{3}$$

$$\boldsymbol{S} = \left(\sin\frac{\pi}{7},\, \sin\frac{2\pi}{7},\, \sin\frac{3\pi}{7} \right) \tag{4}$$

$$\lambda_k = \boldsymbol{\sigma}_k \cdot \boldsymbol{S} \quad (k\text{=1,2,...}) \tag{5}$$

Then

$$\lambda_{k+1} = \left(1 + 2\cos\frac{\pi}{7} \right) \lambda_k \quad (k\text{=1,2,...}) \tag{6}$$

Proof:

$$\lambda_{k+1} = \boldsymbol{\sigma}_{k+1} \cdot \boldsymbol{S} \tag{7}$$

$$= \boldsymbol{E}_3 \boldsymbol{\sigma}_k^{\mathsf{T}} \cdot \boldsymbol{S} \tag{8}$$

$$= \left(\sigma_k(1) + \sigma_k(2),\; \sigma_k(1) + \sigma_k(2) + \sigma_k(3), \sigma_k(2) + 2\sigma_k(3) \right) \tag{9}$$

$$\begin{aligned} &= \left(\sigma_k(1) + \sigma_k(2) \right) \sin\frac{\pi}{7} \\ &+ \left(\sigma_k(1) + \sigma_k(2) + \sigma_k(3) \right) \sin\frac{2\pi}{7} \\ &+ \left(\sigma_k(2) + 2\sigma_k(3) \right) \sin\frac{3\pi}{7} \end{aligned} \tag{10}$$

$$\begin{aligned} &= \left(\sigma_k(1) + \sigma_k(2) \right) \cos\frac{5\pi}{14} \\ &+ \left(\sigma_k(1) + \sigma_k(2) + \sigma_k(3) \right) \cos\frac{3\pi}{14} \\ &+ \left(\sigma_k(2) + 2\sigma_k(3) \right) \cos\frac{\pi}{14} \end{aligned} \tag{11}$$

$$\begin{aligned} &= \sigma_k(1)\cos\frac{5\pi}{14} + \sigma_k(2)\cos\frac{3\pi}{14} + \sigma_k(3)\cos\frac{\pi}{14} \\ &+ \sigma_k(1)\cos\frac{3\pi}{14} + \sigma_k(2)\left(\cos\frac{5\pi}{14} + \cos\frac{\pi}{14} \right) + \sigma_k(3)\left(\cos\frac{3\pi}{14} + \cos\frac{\pi}{14} \right) \end{aligned} \tag{12}$$

$$
\begin{aligned}
&= \sigma_k(1)\cos\frac{5\pi}{14} + \sigma_k(2)\cos\frac{3\pi}{14} + \sigma_k(3)\cos\frac{\pi}{14} \\
&+ \sigma_k(1)\left(2\cos\frac{5\pi}{14}\cos\frac{2\pi}{14}\right) \\
&+ \sigma_k(2)\left(2\cos\frac{3\pi}{14}\cos\frac{2\pi}{14}\right) \\
&+ \sigma_k(3)\left(2\cos\frac{2\pi}{14}\cos\frac{\pi}{14}\right)
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&= \left(\sigma_k(1) + 2\sigma_k(1)\cos\frac{2\pi}{14}\right)\cos\frac{5\pi}{14} \\
&+ \left(\sigma_k(2) + 2\sigma_k(2)\cos\frac{2\pi}{14}\right)\cos\frac{3\pi}{14} \\
&+ \left(\sigma_k(3) + 2\sigma_k(3)\cos\frac{2\pi}{14}\right)\cos\frac{\pi}{14}
\end{aligned}
\tag{14}
$$

$$
= \left(1 + 2\cos\frac{\pi}{7}\right)\left(\sigma_k(1)\cos\frac{5\pi}{14} + \sigma_k(2)\cos\frac{3\pi}{14} + \sigma_k(3)\cos\frac{\pi}{14}\right) \tag{15}
$$

$$
= \left(1 + 2\cos\frac{\pi}{7}\right)\left(\sigma_k(1)\sin\frac{\pi}{7} + \sigma_k(2)\sin\frac{2\pi}{7} + \sigma_k(3)\sin\frac{3\pi}{7}\right) \tag{16}
$$

$$
= \left(1 + 2\cos\frac{\pi}{7}\right)\lambda_k \tag{6}
$$

□