Sum of Sine Identities

The idea is: your algorithm does not produce all sum-of-sine identities but it seems to produce a spanning set for them.

The case $n = 15$.

The cyclotomic polynomial of order 30 is:

$$Q_{30} = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1.$$

So the minimal polynomial of $2\cos(\pi/15)$ is

$$\Psi(Q_{30}) = -1 - D_1 + D_3 + D_4 = x^4 + x^3 - 4x^2 - 4x + 1.$$

Call this m_{15}. To get sum-of-sine identities, involving at most $7\pi/15$, we want multiples of m_{15} of degree at most 6. So we consider $(ax^2 + bx + c)m_{15}$. Write this as a linear combination of Dickson polynomials of the second kind to get

$$aE_6 + (a+b)E_5 + (a+b+c)E_4 + cE_3 - (2a+b+c)E_2 - (3a+ab+ac)E_1 - (2a+2b+c)E_0.$$

We take the obvious basis. When $a = 1, b = 0, c = 0$ we get

$$E_6 + E_5 + E_4 - 2E_2 - 3E_1 - 2E_0.$$

Plugging in $2\cos(\pi/15)$ gives the identity:

$$\sin 7\theta + \sin 6\theta + \sin 5\theta - 2\sin 3\theta - 3\sin 2\theta - 2\sin\theta = 0,$$

where $\theta = \pi/15$. Recall that $E_k(2\cos\theta)\sin\theta = \sin(k+1)\theta$, so that the multiple of θ is one more than the index of E. I will abbreviate this to

$$B_1 = (7, 6, 5, -3, -3, -2, -2, -2, -1, -1).$$

When $a = 0, b = 1, c = 0$ we get $E_5 + E_4 - E_2 - 2E_1 - 2E_0$ and so the identity

$$B_2 = (6, 5, -3, -2, -2, -1, -1).$$

Lastly, $a = 0, b = 0, c = 1$ gives $E_4 + E_3 - E_2 - 2E_1 + E_0$ and the identity

$$B_3 = (5, 4, -3, -2, -2, -1).$$

As worked out in your notes, your algorithm gives three identities:

$$A_1 = (7, -5, -4, 2, 1)$$
$$A_2 = (7, -3, -2)$$
$$A_3 = (6, -4, -1).$$

These clearly do not account for all identities. But they are a basis:

$$A_1 = B_1 - B_2 - B_3 \qquad A_2 = B_1 - B_2 \qquad A_3 = B_2 - B_3.$$

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The case $n = 18$.

Here

$$\begin{aligned} Q_{36} &= x^{12} - x^6 + 1 \\ m_{18} &= \Psi(Q_{36}) = -1 + D_6 \\ &= x^6 - 6x^4 + 9x^2 - 3. \end{aligned}$$

We want multiples of m_{18} of degree at most 8. We take $(ax^2 + bx + c)m_{18}$ and write this as a linear combination of E's. We get

$$aE_8 + bE_7 + (a+c)E_6 - (a+c)E4 - bE_3 - 2aE_2 - bE_1 - (a+c)E_0.$$

The standard basis gives:

$$\begin{aligned} (a,b,c) &= (1,0,0) \qquad & B_1 &= (9,7,-5,-3,-3,-1) \\ &= (0,1,0) \qquad & B_2 &= (8,-4,-2) \\ &= (0,0,1) \qquad & B_3 &= (7,-5,-1). \end{aligned}$$

Your algorithm gives

$$\begin{aligned} &\text{for } 3 \times 6 \qquad & A_1 &= (7,-5,-1) \\ & & A_2 &= (8,-4,-2) \\ & & A_3 &= (9,-3,-3) \\ &\text{for } 9 \times 2 \qquad & A_4 &= (9,-7,-7,5,5,-3,-3,1,1). \end{aligned}$$

The A's do not form a basis since $A_4 = A_3 - 2A_1$. But they contain a basis since

$$A_1 = B_3 \qquad A_2 = B_2 \qquad A_3 = B_1 - B_3.$$

So again the A's span all the sum-of-sine identities.