3.9 Partitioning ROMBIX-2n into Ovals

The following fundamental relation exists between Ovals and triangular numbers:

THEOREM 3.8.1
If S is a Strictly Convex Oval tiled by p rhombs of SRI_{2n}, then p is a triangular number $t(r) = r(r+1)/2$, ... (r=0, 1, 2, ..., n-1); r is called the *rank* of $t(r)$. If we denote the number of edges of S by $2g(r)$, then $g(r) = r+1$.

As described in the ROMBIX-16 instructions, the sixteen rombiks of ROMBIX-16 can be distributed among tilings of 3, 4, 5, 6, 7, or 8 Strictly Convex Ovals. A partition into *two* Ovals is impossible, because 28, the number of rhombs in SRI_{16}, cannot be partitioned into two triangular numbers*. It is possible, however, to partition the ROMBIX-16 set into two *congruent Stretched Ovals*, each of which can be tiled with exactly two monochrome subsets whose common boundary is of the same shape in each of the Ovals.

As a consequence of Theorem 3.8.1, a partition of SRI_{2n} or of ROMBIX-2n into a set of Ovals implies that the triangular number $\binom{n}{2}$ can be expressed as the sum of a set of smaller triangular numbers. A search by computer suggests the following conjecture:

CONJECTURE 3.8.1
Let $t(r)$ be any triangular number of rank $r \geq 4$.
Then for every $k \in [3, r+1]$, $t(r)$ can be expressed as the sum of k non-zero triangular numbers, which are not necessarily distinct.

Table 3.8.1 shows examples of partitions of the triangular numbers 10, 15, 21, 28, 36, and 45.

If Conjecture 3.8.1 is true, then it may also be true that for $n \geq 5$ the rhombs in SRI_{2n} can be partitioned into k Strictly Convex Ovals tiled by rhombs, for every $k \in [3, n]$ (since $n=r+1$). Any such partition requires not only the existence of a partition of the triangular number $\binom{n}{2}$ into a set of smaller triangular numbers, but also the existence of a set of Ovals whose combined rhombic inventories are equal to the set of rhombs in SRI_{2n}.

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* It is a well-known theorem [Beiler 1966] that every integer is either a triangular number, the sum of two triangular numbers, or the sum of three triangular numbers. A computer search reveals that 7 of the first 15 triangular numbers, 64 of the first 100 triangular numbers, and 775 of the first 1000 triangular numbers can be partitioned into two triangular numbers.

Table 3.8.1

Examples of partitions of triangular numbers $t(r) = r(r+1)/2$ into k smaller triangular numbers ($3 \le k \le r+1$) for $4 \le r \le 9$

r	$t(r)$	k = 3	4	5	6	7	8	9	10
		Partition							
4	10	1+3+6	1+3+3+3	1+1+1+1+6					
5	15	3+6+6	1+1+3+10	1+1+1+6+6	1+1+1+3+3+6				
6	21	1+10+10	3+6+6+6	3+3+3+6+6	1+1+1+6+6+6	1+1+1+3+3+6+6			
7	28	3+10+15	6+6+6+10	3+3+6+6+10	1+1+3+3+10+10	1+3+3+3+6+6+6	1+1+1+1+6+6+6+6		
8	36	6+15+15	6+10+10+10	3+3+10+10+10	1+1+3+6+10+15	1+1+3+3+3+10+15	1+3+3+3+6+10+10	1+1+1+1+3+3+6+10+10	
9	45	15+15+15	10+10+10+15	1+3+10+10+21	1+3+6+10+10+15	3+3+3+6+10+10+10	1+1+1+6+6+10+10+10	1+3+3+3+3+6+6+10+10	3+3+3+3+3+6+6+6+6+6

Even if a particular set S of Ovals can be tiled by the rhombs of SRI_{2n}, it does not follow that S can necessarily be tiled by the rombiks of ROMBIX-2n. For example, if the number of rhombs in every Oval of S is odd, then each of the Ovals must contain an odd number of keystones. Hence the number m of Ovals in S cannot exceed the number of keystones in ROMBIX-2n: $m \le \lfloor n/2 \rfloor$.

We will call an Oval *even* if it contains an even number of rhombs, and *odd* otherwise. Searching for a partition of a standard set of rombiks among a set S of Ovals is easiest when most or all of the Ovals are even. In such cases, one first tiles as many of the Ovals as possible with twin rombiks only, and then the keystones are used—sparingly—in the tilings of the remaining Ovals.

Let us consider the following restricted class of partitions of SRI_{2n} into Ovals:

Each of the m Ovals in the set S contains the same number τ of rhombs.

We define the triangular number $\binom{n}{2}$ to be *isopartite of order m* if $\binom{n}{2}$ is divisible by a triangular number τ:

$$\binom{n}{2} = m\,\tau \quad (n, m>1).$$

We will call two Ovals *homologous* if each contains τ rhombs from SRI_{2n}, and we will call a set S of m Ovals *a complete homologous set* if the $\binom{n}{2}$ rhombs of SRI_{2n} can be partitioned among the Ovals in the set. We conjecture that for every triangular number $\binom{n}{2}$ that is *isopartite of order m*, there exists at least one complete homologous set of m Ovals tiled by the rhombs of SRI_{2n}.

Let us examine how the *parity* of τ (*i.e.*, whether τ is odd or even) restricts the partitioning of a standard set of rombiks among a set S of m homologous Ovals of order n.

If all of the Ovals in S are even, then it is possible that most of them can be tiled by twin rombiks. In such cases, we cannot rule out the possibility that a standard set of rombiks can be partitioned among the Ovals of S. Whether such a partition exists in a particular case can be determined by a search.

If every Oval in S is odd, then—as mentioned above—there must be at least one keystone contained in each Oval, and the number m of Ovals in S cannot be larger than the number of keystones in ROMBIX-2n: $m \le \lfloor n/2 \rfloor$.

Consider the following examples of homologous sets of odd Ovals tiled by rombiks:

(a) τ=3 (g=3)

The only value of n for which a complete homologous set of 3-Ovals can be tiled by a standard set of rombiks is *four*. (The two homologous 3-Ovals in this case are congruent.) For every value of $n>4$ for which $\binom{n}{2}$ is divisible by three, *viz.*, for $n = 6, 7, 9, 10, 12, 13, 15, 16, \ldots$, the number $\lfloor n/2 \rfloor$ of keystones in ROMBIX-$2n$ is less than the number m of 3-Ovals, and therefore no partition of the rombiks among the 3-Ovals is possible.

(b) τ=15 (g=6)

The only values of n which are candidates for the tiling of a complete homologous set of 6-Ovals by a standard set of rombiks are 10, 15, 16, and 21. For every value of $n>21$ for which $\binom{n}{2}$ is divisible by fifteen, *viz.*, $n = 25, 30, 31, 36, 40, \ldots$, the number $\lfloor n/2 \rfloor$ of keystones is less than the number m of 6-Ovals, and therefore no partition of the rombiks among the 6-Ovals is possible.

Let us consider a specific example of case (b) for n=10. There are 45 rhombs in SRI_{20}, and there are sixteen different shapes of 6-Ovals for n=10. A computer search reveals that there are six ways to select three of these sixteen 6-Ovals—allowing repetitions—whose total rhombic inventory is equal to that of SRI_{20}. Eleven of the sixteen Ovals occur in these six partitions.

The sixteen 6-Ovals for n=10 are listed below, together with their rhombic inventory vectors. Each of the eleven starred Ovals that appears in *one* three-Oval partition is starred once; each Oval that appears in *two* such partitions is starred twice.

Oval number	TAIS	Rhombic Inventory Vector
1	[511111]	(5 4 3 2 1)
2	[312121]	(3 2 5 4 1)
3	[222211]	(2 5 2 5 1)
4**	[331111]	(4 3 4 3 1)
5*	[321112]	(3 4 4 2 2)
6**	[411211]	(4 3 2 4 2)
7**	[222121]	(2 4 4 3 2)
8*	[313111]	(4 2 3 4 2)
9	[311311]	(4 2 2 4 3)
10	[221221]	(2 4 4 2 3)
11**	[421111]	(4 4 3 3 1)
12**	[321121]	(3 3 4 4 1)
13*	[412111]	(4 3 3 3 2)
14**	[322111]	(3 4 3 3 2)
15**	[312211]	(3 3 3 4 2)
16*	[321211]	(3 3 4 3 2)

On p. 22 are shown the eleven starred Ovals tiled by rhombs of SRI_{20}, together with a tiling of the complete homologous set 4-6-7 by rombiks. The other five homologous sets of three 6-Ovals for n=10 are as follows:

4-14-15 5-6-12 7-8-11 11-15-16 12-13-14.

On p. 23 is shown a partition of the rombiks of ROMBIX-32 among twenty 4-Ovals (τ=6).

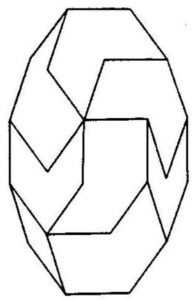
On p. 24 is shown a partition of the rombiks of ROMBIX-32 among twelve 5-Ovals (τ=10).

Suggested problem:

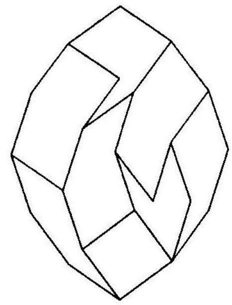
Search for a partition of the twenty rombiks of ROMBIX-18 among six 4-Ovals (τ=6). There are altogether ten 4-Ovals for n=9. A computer search reveals that there are thirty-five ways to partition the rhombs of SRI_{18} among six 4-Ovals. In these thirty-five partitions, four of the ten 4-Ovals occur twelve times, and six of the ten 4-Ovals occur twenty-seven times.

PARTITION OF THE 25 ROMBIKS OF ROMBIX-20 AMONG 3 6-OVALS

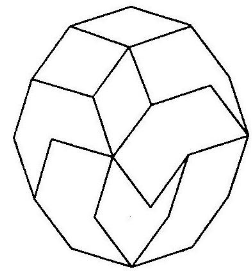
Each 6-Oval is identified by its TAIS (Turning Angle Index Sequence)



[331111]

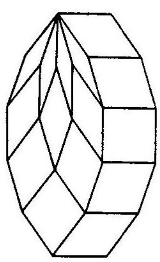


[411211]

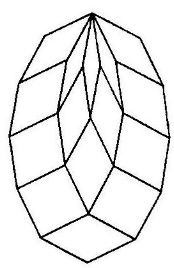


[222121]

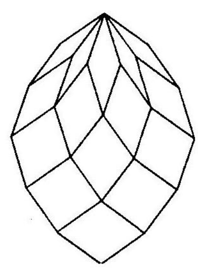
THE ELEVEN 6-OVALS THAT OCCUR IN HOMOLOGOUS THREE-OVAL SETS (n=10)



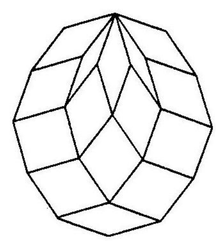
[331111]



[321112]



[411211]



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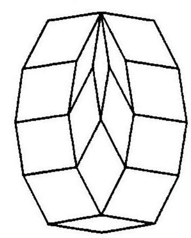
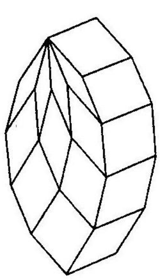
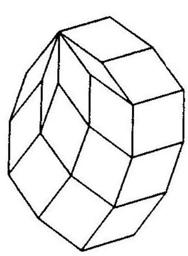


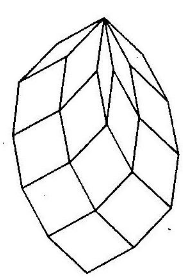
[313111]



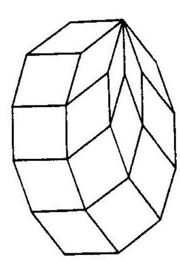
[421111]



[321121]



[412111]



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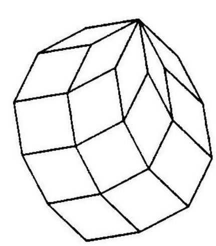
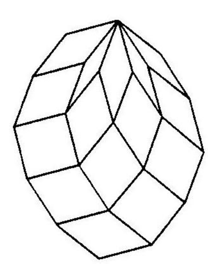


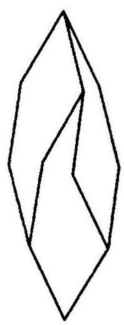
[312211]

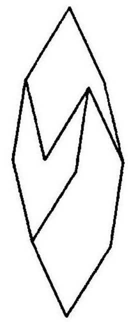


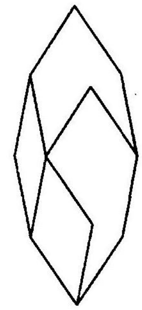
[321211]

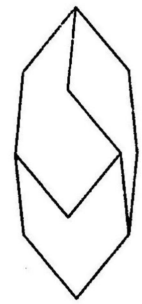
PARTITION OF THE 64 ROMBIKS OF ROMBIX-32 AMONG 20 4-OVALS

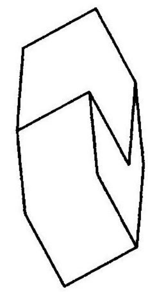
Each 4-Oval is identified by its TAIS (Turning Angle Index Sequence).


[11 2 2 1]


[11 2 2 1]


[10 2 2 2]


[9 3 1 3]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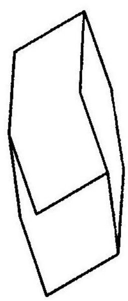
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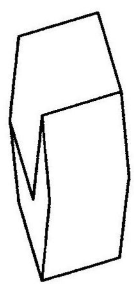
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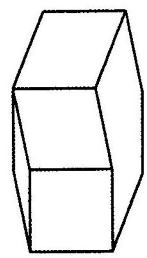
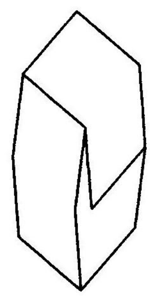

[7 6 2 1]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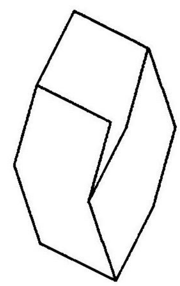
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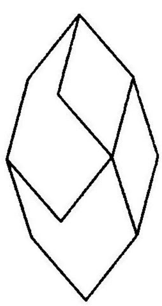
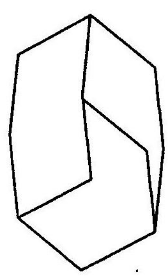
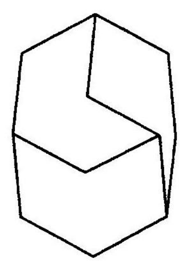
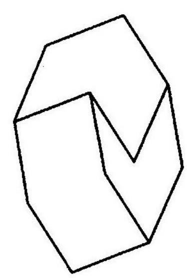

[9 2 3 2]


[6 5 1 4]


[5 5 1 5]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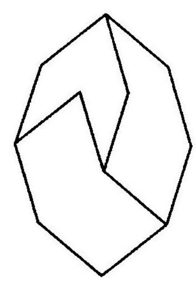
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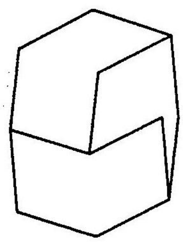
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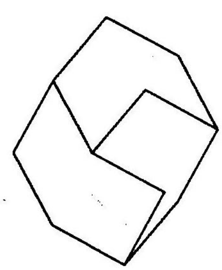
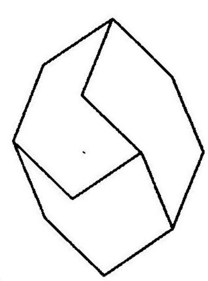

[7 1 5 3]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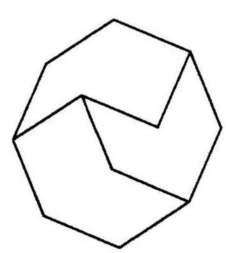
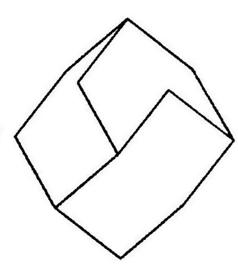
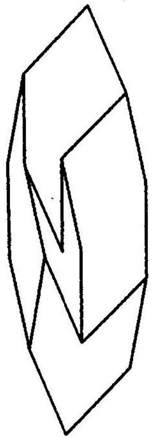

[5 3 4 4]

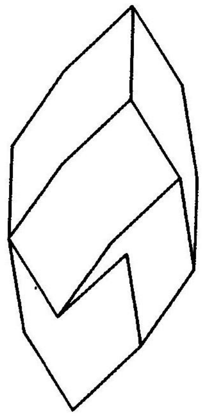

[7 1 6 2]

PARTITION OF THE 64 ROMBIKS OF ROMBIX-32 AMONG 12 5-OVALS

Each 5-Oval is identified by its TAIS (Turning Angle Index Sequence).



[10 3 1 1 1]



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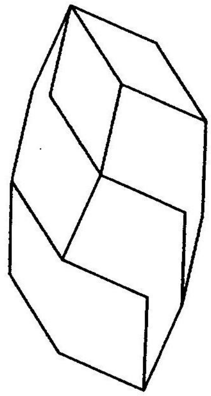
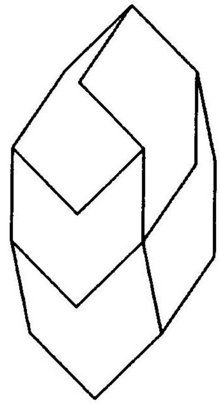


[5 1 5 2 3]



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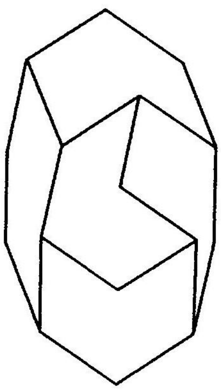

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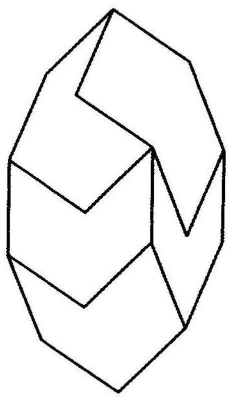

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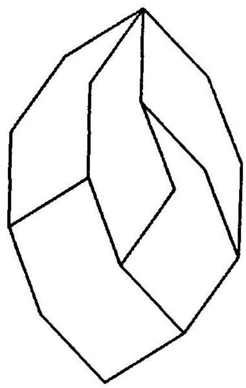

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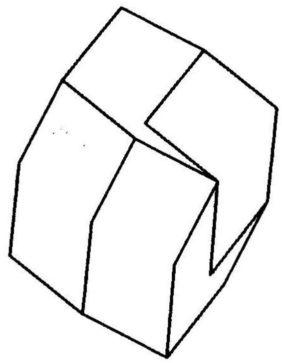

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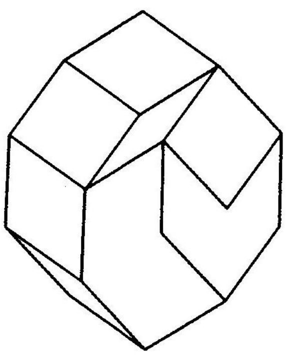

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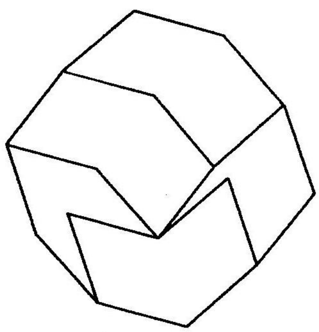

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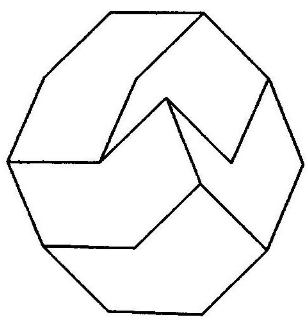
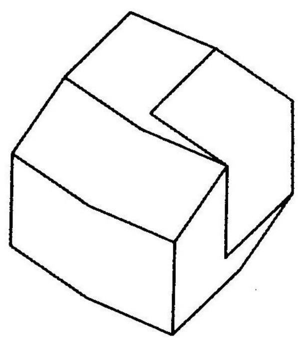


[4 4 2 4 2]



[6 1 3 5 1]