1. Sine expansion of tangent.

Let $n = 2m + 1$ be odd. The system of equations:

$$\sigma_2(n) = 1$$

$$\begin{aligned} \sigma_2(n) + \sigma_4(n) &= 0 \\ \sigma_4(n) + \sigma_6(n) &= 0 \\ \sigma_6(n) + \sigma_8(n) &= 0 \\ &\vdots \\ \sigma_{m-1} + \sigma_m(n) &= 0 \end{aligned}$$

$$\begin{aligned} \sigma_1(n) + \sigma_3(n) &= 0 \\ \sigma_3(n) + \sigma_5(n) &= 0 \\ \sigma_5(n) + \sigma_7(n) &= 0 \\ &\vdots \end{aligned}$$

has the solutions:

(a) if $n \equiv 1 \pmod 4$ then

$$\begin{aligned} \sigma_{2\ell}(n) &= (-1)^{\ell+1} \quad \text{for} \quad 1 \le \ell \le (n-1)/4 \\ \sigma_{2\ell+1}(n) &= (-1)^{\ell+1} \quad \text{for} \quad 0 \le \ell \le (n-5)/4. \end{aligned}$$

(b) if $n \equiv 3 \pmod 4$ then

$$\begin{aligned} \sigma_{2\ell}(n) &= (-1)^{\ell+1} \quad \text{for} \quad 1 \le \ell \le (n-3)/4 \\ \sigma_{2\ell+1}(n) &= (-1)^{\ell} \qquad \text{for} \quad 0 \le \ell \le (n-3)/4. \end{aligned}$$

Theorem. *For $\sigma_k(n)$ as above:*

$$\frac{1}{2}\tan(\pi/n) = \sum_{k=1}^{m} \sigma_k(n) \sin(k\pi/n).$$

Proof. Let $w = e^{\pi i/n}$ and let A be the right-hand side of the identity. Then

$$2iA = \sum_{k=1}^{m} \sigma_k(n)(w^k - w^{-k}).$$

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Multiply by $w + w^{-1}$:

$$
\begin{aligned}
(w+w^{-1})2iA &= \sum_{k=1}^{m} \sigma_k(n)(w^{k+1} + w^{k-1} - w^{-(k-1)} - w^{-(k+1)}) \\
&= \frac{1}{w^{m+1}} \sum_{k=1}^{m} \sigma_k(n)(w^{k+m+2} + w^{k+m} - w^{m-k+2} - w^{m-k}) \\
&= \frac{1}{w^{m+1}} \Bigg[\sigma_m(n)w^{n+1} + \sigma_{m-1}(n)w^n + \sum_{k=1}^{m-2} (\sigma_k(n) + \sigma_{k+2}(n))w^{m+k+2} \\
&+ \sigma_2(n)(w^{m+2} - w^m) \\
&- \sum_{k=1}^{m-2} (\sigma_k(n) + \sigma_{k+2}(n))w^{m-k} - \sigma_{m-1}(n)w - \sigma_m(n) \Bigg] \\
&= [\sigma_m(n)w^{n+1} + \sigma_{m-1}(n)w^n + w^{m+2} - w^m - \sigma_{m-1}(n)w - \sigma_m(n)]/w^{m+1}
\end{aligned}
$$

where we used the system of equations given above. Now $w^n = -1$, $w^{n+1} = -w$ and $\sigma_{m-1}(n) + \sigma_m(n) = 0$ so that

$$\sigma_m(n)w^{n+1} + \sigma_{m-1}(n)w^n - \sigma_{m-1}(n)w - \sigma_m(n) = 0.$$

Hence

$$(w + w^{-1})2iA = (w^{m+2} - w^m)/w^{m+1} = w - w^{-1},$$

and so

$$A = \frac{1}{2i} \frac{w - w^{-1}}{w + w^{-1}} = \frac{1}{2} \tan(\pi/n).$$

□

2. A formula for the minimal polynomial of $2\cos(2\pi/r)$.

Notice the two extra 2's in $\mathbf{2}\cos(\mathbf{2}\pi/r)$.

It is simpler (for me) to work with the *Dickson polynomials* $D_n(x) = 2T_n(x/2)$, where T_n is the Chebyshev polynomial of the first kind (i.e. $\cos(n\theta) = T_n(\cos\theta)$). The Dickson polynomials may be quickly computed from

$$
\begin{aligned}
D_0(x) &= 2 \\
D_1(x) &= x \\
D_{n+2}(x) &= xD_{n+1}(x) - D_n(x).
\end{aligned}
$$

The Dickson polynomials have some algebraic advantages: they are monic (leading coefficient is 1) and satisfy

$$D_n(x + x^{-1}) = x^n + x^{-n}.$$

In fact, $D_n(x)$ is the unique monic polynomial of degree n that satisfies this identity.

I will also use the cyclotomic polynomials $Q_n(x)$, the minimal polynomials of $e^{2\pi i/n}$. The roots of $Q_n(x)$ are precisely the primitive nth roots of unity. If w is a primitive nth root of unity then so is $1/w$, and so $1/w$ is also a root. This means that $Q_n(x)$ is *self-reciprocal* ,that is, has the form:

$$a_0x^{2d}+a_1x^{2d-1}+\cdots+a_{d-2}x^{d+2}+a_{d-1}x^{d+1}+a_dx^d+a_{d-1}x^{d-1}+a_{d-2}x^{d-2}+\cdots+a_1x+a_0,$$

where $d=\varphi(n)/2$. Note that $a_0=1$.

Theorem. *Suppose $r\geq 3$. Set $d=\varphi(r)/2$. Write*

$$Q_r(x)=a_dx^d+\sum_{j=0}^{d-1}a_j(x^{2d-j}+x^j).$$

Then the minimal polynomial of $2\cos(2\pi/r)$ is

$$P_r^D(x)=a_d+\sum_{j=0}^{d-1}a_jD_{d-j}(x).$$

Proof. Set $w=e^{2\pi i/r}$. Compute:

$$\begin{aligned}
w^dP_r^D(w+w^{-1}) &= w^d\left[a_d+\sum_{j=0}^{d-1}a_jD_{d-j}(w+w^{-1})\right]\\
&= w^d\left[a_d+\sum_{j=0}^{d-1}a_j(w^{d-j}+w^{-(d-j)})\right]\\
&= a_dw^d+\sum_{j=0}^{d-1}a_j(w^{2d-j}+w^j)\\
&= Q_r(w)=0.
\end{aligned}$$

The leading term of P_r^D comes form the $j=0$ term, $a_0D_d(x)=x^d+\cdots$. In particular, P_r^D is monic of degree d and hence the minimal polynomial of $w+w^{-1}=2\cos(2\pi/r)$. □

As a simple example (using MAPLE):

$$Q_{60}(x)=x^{16}+x^{14}-x^{10}-x^8-x^6+x^2+1.$$

Thus the minimal polynomial of $2\cos(2\pi/60)$ is

$$P_{60}^D(x)=-1+D_{8-0}(x)+D_{8-2}(x)-D_{8-6}(x).$$

And

$$\begin{aligned}D_2(x) &= x^2 - 2\\ D_6(x) &= x^6 - 6x^4 + 9x^2 - 2\\ D_8(x) &= x^8 - 8x^6 + 20x^4 - 16x^2 + 2.\end{aligned}$$

Hence

$$P_{60}^D(x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1.$$

To get a minimal polynomial of $\cos(2\pi/r)$ we take:

$$\begin{aligned}P_r^T(x) = P_r^D(2x) &= a_d + \sum_{j=0}^{d-1} a_j D_{d-j}(2x)\\ &= a_d + 2\sum_{j=0}^{d-1} a_j T_{d-j}(x).\end{aligned}$$

Again $T_n(x)$ is the Chebyshev polynomial of the first kind and we used $D_n(2x) = 2T_n(x)$. P_r^T is not monic so it is (technically) not the minimal polynomial of $\cos(2\pi/r)$, but a scalar multiple of the minimal polynomial.

Lastly, to get a (scalar multiple) of the minimal polynomial of $\cos(\pi/n)$ simply take P_r^T for $r = 2n$. It helps to know that, if n is odd, then $Q_{2n}(x) = Q_n(-x)$.

3. The case n is prime.

When n is an odd prime we get two different expressions for the minimal polynomial of $\cos(\pi/n)$.

We use the Chebyshev polynomials of the second kind $U_t(x)$, where $\sin((t+1)\theta) = U_t(\cos\theta)\sin\theta$. The sine expansion of tangent becomes (here $n = 2m+1$):

$$\begin{aligned}\frac{1}{2}\tan(\pi/n) &= \sum_{k=1}^{m} \sigma_k(n)\sin(k\pi/n)\\ \frac{1}{2}\frac{\sin\pi/n}{\cos\pi/n} &= \sum_{k=1}^{m} \sigma_k(n) U_{k-1}(\cos\pi/n)\sin\pi/n\\ 1 &= 2\cos(\pi/n)\sum_{k=1}^{m} \sigma_k(n) U_{k-1}(\cos\pi/n).\end{aligned}$$

Hence $\cos(\pi/n)$ is a root of

$$F_n^1(x) = -1 + 2x\sum_{k=1}^{m} \sigma_k(n) U_{k-1}(x).$$

As $\deg F_n^1 = m = (n-1)/2$, this is (a scalar multiple of) the minimal polynomial.

For the second formula,

$$Q_{2n}(x) = Q_n(-x) = \sum_{j=0}^{n-1} (-1)^j x^j.$$

Hence $\cos(\pi/n)$ is a root of

$$F_n^2(x) = (-1)^m + 2\sum_{j=1}^{m} (-1)^{m-j} T_j(x).$$

When $n \equiv 3, 5 \pmod 8$, $F_n^1(x) = F_n^2(x)$ and when $n \equiv \pm 1 \pmod 8$ then $F_n^1(x) = -F_n^2(x)$.

As a simple example:

$$\begin{aligned} F_{13}^1 &= -1 + 2x(-U_0 + U_1 + U_2 - U_3 - U_4 + U_5) \\ &= 64x^6 - 32x^5 - 80x^4 + 32x^3 + 24x^2 - 6x - 1 \end{aligned}$$

$$\begin{aligned} F_{13}^2 &= 1 + 2(-T_1 + T_2 - T_3 + T_4 - T_5 + T_6) \\ &= 64x^6 - 32x^5 - 80x^4 + 32x^3 + 24x^2 - 6x - 1. \end{aligned}$$

This equality can (probably) be proven directly. For instance, for any n (prime or not)

$$x(U_{n-1} - U_n - U_{n+1} + U_{n+2}) = -T_n + T_{n+1} - T_{n+2} + T_{n+3}.$$