The Geometry of Minimal Surfaces

It is very easy to construct soap film surfaces, but surprisingly difficult to describe them mathematically. In recent years, it has been possible with the help of computer graphics to discover many new minimal surfaces and to understand their geometry better. This is a new contribution to a 200 year-old problem.

Hermann Karcher and Konrad Polthier
(rough translation by Alan Schoen, 1996)

The most natural introduction to minimal surfaces is by means of soap films. If we dip a wire loop of any shape into a soap solution and withdraw it carefully, a very thin iridescent film is formed in the loop. Because of its surface tension, the soap film assumes a shape in which its potential energy is a minimum and it therefore finds itself in stable equilibrium. Since this energy is proportional to the surface area, the soap film has a smaller surface area than every other possible surface that can be spanned by the same wire loop. For that reason we mathematicians call such surfaces minimal surfaces.

The name minimal surface is to be sure somewhat misleading, because the standard definition today does not refer to the surface area, but rather to curvature properties of the surface. All surfaces that physically correspond to unstable equilibrium positions of the potential energy are also called minimal surfaces. A surface that has the required curvature properties is not necessarily the surface of least area among those that fit a given boundary curve.

(The ordinary soap bubble is not a minimal surface, because the enclosed air is under extra pressure and therefore the potential energy is a minimum under entirely different constraints.)

The versatile mathematician Joseph-Louis de Lagrange (1736-1813), Professor at Turin, Berlin, and Paris, in 1760 opened up the mathematical investigation of minimal surfaces with the question:

> Does there exist for every arbitrarily complicated boundary curve a surface of least area?

In spite of the fact that this problem appears easily solvable by experiments with soap films, for a long time all attempts at a mathematical treatment were disappointing. Lagrange himself indeed discovered very quickly the conditions—so-called partial differential equations—that a minimal surface must satisfy at every point, but using the theoretical methods of that time, it was quite hopeless to try to answer his question. And so only a single less interesting solution to these equations was known: namely, the surface obtained when the boundary curve lies in a plane.

The French geometer and engineer Jean Baptiste Meusnier (1754-1793) furnished the first non-trivial examples of minimal surfaces: the catenoid (also called the chain surface, because it is the surface swept out when one rotates the catenary curve that corresponds to a freely hanging chain about a suitable horizontal line) and the helicoid, or screw surface. The next example was first published in 1835 and was regarded as so sensational that the Paris Academy distinguished its discoverer Heinrich Ferdinand Scherk (1798-1885), Professor at Kiel and Bremen, with a large prize.

In the meantime, as a result of insightful experiments, understanding progressed rapidly. The Belgian physicist Joseph Antione Ferdinand Plateau (1801-1883), a Professor at Ghent, carried out extensive experiments with soap films in the mid-nineteenth century. Data from his accurate measurments of many surfaces served later as verification of theoretical results. Out of his investigations there developed the mathematically important conjecture that every closed boundary curve that neither touches itself nor intersects itself can be spanned by a minimal surface. The problem of finding this surface for a given boundary curve entered mathematical history as the "Plateau problem".

Complete mathematical proof of its solvability—and therewith a partial answer to the question of Lagrange—was furnished first in 1931 by the American Jesse Douglas (1897-1965) and by the Hungarian Tibor Rado (1895-1965), independently of one another, using completely newly developed mathematical tools. Their result marked a brilliant high point in the calculus of variations, a mathematical discipline whose development for more than 200 years has been tightly bound up with work on the question of Lagrange. In 1936 Douglas obtained the Fields medal for his work, which is the highest distinction awarded by the Inernational Mathematical Union. As has happened so frequently in the history of mathematics, the solution of this long open problem immediately generated new questions. The theorem proved by Douglas and Rado proved only the mathematical *existence* of a minimal surface for a given boundary curve, but said very little about its geometrical properties.

Further development concentrated therefore on a better understanding of this solution of the Plateau problem, in particular its behavior close to the boundary curve. In the last thirty years, interest has concentrated on a class of minimal surfaces that are conceptually far removed from the original meaning of the term minimal surfaces: they have no boundary curve—on the contrary, they are infinitely extended. Because there were no sufficiently complicated examples for exact investigation, this new development proceeded only slowly.

New surfaces and computer graphics

In recent years it has become possible to enlarge this family considerably. We have learned how to construct new unbounded surfaces aimed at chosen geometrical shapes and also how to alter known surfaces.

These new examples and especially their computer graphics images have in turn led to a better understanding of the geometry of minimal surfaces generally and their underlying principles. They have also called attention to the structure of these surfaces in other scientific disciplines like crystallography and polymer chemistry. And not only do the images have obvious esthetic charm, but also they serve as intuitive material for teaching and as aids to understanding for scholars in the discussion of new phenomena.

We want now to explain several geometrical properties of minimal surfaces—in particular with the objective of investigating every infinitely extended surface that nowhere intersects itself (in technical language, one calls such surfaces 'embedded').

Let us examine area-minimizing surfaces a little more closely. Obviously the outer surface of every mountaintop is larger than the surface on the ground underneath the mountain, and therefore the surface area would be reduced if the mountain were removed. Meusnier, pursuing this argument—that a minimal surface cannot exhibit any domes but on the contrary must at every point be curved toward both sides—concluded that in a small neighborhood of every point, the surface must resemble a horse's saddle, which curves upward in the longitudinal direction and downward in the transverse direction. An exact analysis shows that both curvatures must be of equal magnitude.

In order to make these curvature properties more precise, we must first clarify what we mean by the curvature of a plane curve. A large circle is more weakly curved than a small circle. The mathematical definition corresponds to this intuitively obvious representation of curvature. A circle of radius r is assigned a curvature equal to $1/r$. A straight line, considered as an infinitely large circle, has curvature zero.

A circle is equally curved at every one of its points. On the other hand, for an arbitrary curve, the curvature varies from point to point, and yet we are led back to the circle. Let us consider a car driving along such a curve.

The positions of the wheels of the car at every point determine a circle on which the car would continue to travel—the so-called circle of curvature, if the driver did not continuously change the position of the steering wheel. The curvature at any point on the curve is defined as the curvature of this circle.

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Fig. 3. The curvature at a point on a curve is defined with the help of the so-called circle of curvature. This is the circle that most closely conforms to the curve at the point. If one drives a car along the curve and the steering wheel is instantaneously held fixed, then the car will continue along the circle of curvature. A circle of curvature can either (a) touch the curve along the inside of the curve, (b) touch the curve along the outside of the curve, or (c) intersect the curve. While the first two cases occur only at points of either maximum or minimum curvature, the third case is the general rule—even though this is difficult to recognize from a drawing of a curve.

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the tangent planes at points along the lines of symmetry. Although our eyes cannot distinguish such a minimal surface in the neighborhood of this near-circle from the surface in the region near the waist of a catenoid, the two surfaces look completely different away from this region.

On account of this sensitivity, our pictures are misleading. They all leave the viewer with no doubt about how the surface will continue outside of the illustrated portion of the surface. This, however, is contradictory to the nature of minimal surfaces: to every picture there corresponds an enormous number of minimal surfaces that appear indistinguishable within the illustrated piece of the surface, but that sufficiently far away remain quite wild and unimaginable.

Controlled alteration of surfaces

Requisite control of minimal surfaces out to infinity was developed in the work of Osserman on the Weierstrass representation. It succeeded in a very clear way in changing even already known surfaces, making them more complicated, even though this change has a marked effect on only a small part of the surface. This success appears to contradict the above described 'inheritability' property of minimal surfaces, which usually react in the form of catastrophic perturbations at a great distance from a place where small changes are made. Mathematicians did not have to wait for an abundance of new examples of this effect.

Luquesio P. Jorge of the Mathematical Research Institute IMPA in Rio de Janeiro and William M. Meeks III, now at the University of Massachusetts at Amherst, were the first who—starting from the approximate complete representation of the surface to be newly constructed—undertook the controlled alteration of a minimal surface out to infinity. Out of the two funnel-shaped ends into which the catenoid spreads out, they made three ends. From this approximate representation, along with Osserman's theory, they concluded in 1980 with the Weierstrass representation formula for a new minimal surface: the *trinoid, in which three ends are fitted together* (cf. Fig. 6a). Today we can build new ends at arbitrary places on the catenoid. The *bidenoid* has two additional tiny ends on the lower portion of the catenoid.

These new complete minimal surfaces are however not embedded, since they intersect themselves, as one can see as soon as one observes a larger portion oi the minimal surface. The ends of the trinoid are mutually interpenetrating. Also the little ends of the bidenoid cut the remaining surface.

In the case of embedded minimal surfaces, on the one hand, until a few years ago periodic surfaces like the helicoid, as well as the Scherk surface and the surfaces of Schwarz were known, but on the other hand, only two non-periodic surfaces—the plane and the catenoid—were known. There was a conjecture that these two were indeed the only two such surfaces. In any case, nobody was able to suggest what geometrical form a candidate surface might have.

> Two complete minimal surfaces that are contiguous to the same curve must be the same everywhere.

"Contiguous" here means not only that the two surfaces have the same curve in common, but also that their tangent planes are the same at every point of the curve. We use the intuitive word "strip" to designate the curve together with the tangent planes of the surface along the curve.

In the case of the catenoid, for example, the circle in the symmetry plane of the surface, together with the tangent planes that are perpendicular to this plane, define such a strip. It follows that a minimal surface that contains a circle and whose tangent planes at points of the circle are perpendicular to the plane of the circle must be the catenoid.

Symmetry properties

Using Björling's theorem, we can prove two important symmetry properties of minimal surfaces. First case: if the curve of the Strip Theorem lies in a plane and all tangent planes of the surface along the curve are perpendicular to this plane, then the strip is mirror-symmetric with respect to the plane of the strip curve. Therefore the surface is continued in the same way on either side of the plane and is therefore mirror-symmetric with respect to the plane. An example of this situation is the aforementioned waist-strip of the catenoid. Second case: a further symmetry property is generated in the same way, if a straight line segment lies in the surface. In that case the strip containing the line segment is reflected in the segment (this is the same as a 180° rotation about the segment), and therefore the complete minimal surface itself must be symmetrical in the same way (*cf.* the straight lines in the helicoid in Fig. 2d).

A piece of a minimal surface can contain several line segments, as is shown by the Gergonne surface, whose boundary curve consists of successively orthogonal line segments. The finite minimal surface bounded by this curve can be reflected at each of these boundary segments and continued beyond them. Through further reflections at the newly created boundary segments the surface can be extended to infinity. If one applies these same symmetry operations (reflection at each of the straight bounday segments) to the square prism from whose edges each boundary segment is composed, infinitely many specimens of the square prism—each incident on another—are generated, with no overlapping. Every prism face borders an empty space of the same shape. As a result of this agreeable property of the symmetry operations, the complete minimal surface is also free of self-intersections.

Unfortunately it is only in especially lucky cases that minimal surfaces can be continued beyond their boundaries in such a convenient way. In particular, it is not true that there exists a minimal surface that contains any arbitrarily chosen strip. On the contrary, such strips are extremely rare.

The distinctions between the shapes of boundary curves that are valid in this context and those that are not are so subtle that they cannot be made visible in pictures. As an illustration of this, one should look at the almost circular lines of symmetry together with

Now we can turn back again to the curvature of a surface—it leads back to the curvature of a plane curve. Let us consider a fixed point on the surface. The plane that best approximates the surface in the immediate vicinity of this point is called the *tangent* plane, and the straight line that is perpendicular to the tangent plane at this point is called the *surface normal*. Every plane that contains the surface normal cuts the surface in a curve, a so-called *normal section*.

Since a minimal surface looks like a saddle in the neighborhood of every one of its points, some normal sections are curved toward one side of the surface and some are curved toward the other. On each side, there is a normal section with maximum curvature; as a rule the planes to which these two normal sections belong are perpendicular.

Minimal surfaces, as we have already observed, are equally curved toward both sides. In other words, at every point the curvatures of the two maximal normal sections are equally large. This equilibrium of the curvatures is probably the basis for the esthetic charm of minimal surfaces.

As we said at the beginning, mathematicians call a surface a minimal surface if at every one of its points it is in curvature equilibrium, and not—as its name precisely suggests—because it has the smallest possible area for a given boundary curve. This characterization has many advantages: for one, it is simpler to examine, since the curvature at every point can be calculated, whereas for the comparison of areas the entire surface, together with surfaces that differ from it very slightly, must be examined. In the second place, the definition is independent of the boundary curve. Consequently, surfaces without a boundary, which extend to infinity, are also minimal surfaces.

Furthermore, minimal surfaces now no longer need to be identified with soap films. Every soap film is indeed a minimal surface, and every sufficiently small piece of a minimal surface can be produced as a soap film in a wire loop that conforms exactly to the shape of the boundary of the piece of surface. But larger pieces can almost never be realized as soap films, since they correspond to unstable equilibria of the potential energy. For example, the two catenoids in Fig. 2c are both in curvature equilibrium, but only the outer catenoid can be generated as a soap film. Obviously the area of the inner catenoid is considerably larger than that of the outer one, and therefore it is not a minimum.

Can one, conversely, represent beyond its boundary a bounded minimal surface as—for example—the solution of a Plateau problem? In general, no. The fact that the curvatures in the neighborhood of the boundary can tend toward infinity prevents continuation. Even if a piece of the surface can be continued further, one must deal with the fact that the continued surface may intersect itself. However, this continuation, if it exists, determines a unique complete minimal surface, no part of which—including every small piece of the surface—can be altered even slightly.

A minimal surface is uniquely determined by something that is even less than a small piece of the surface. We are indebted to the Swedish high school teacher Emmanuel Gabriel Björling (1808-1872) for the so-called Strip Theorem:

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Fig. 6

The *trinoid* (a), constructed by Luquesio P. Jorge and William M. Meeks III, is the first complete minimal surface that was generated from an approximate complete representation of a surface. Jorge and Meeks searched for the representation formula for a surface that consists of three half-catenoids. The *bidenoid* ©, which was found by Hermann Karcher in 1989, implies that the catenoid can be altered in many different ways—in this case, by the addition of two very small catenoids. Neither of these two surfaces is embedded: the open ends of the trinoid, after being extended outward, intersect each other, just as the tiny half-catenoids do on the surface on which they lie. The *Costa surface* (b) is the most famous surface of the new era. From far away it looks like the catenoid together with its equatorial symmetry plane. Nearby it looks like a modification of the equatorial circle of the catenoid and its equatorial plane by a system of tunnels similar to the system of tunnels that modify the straight line segments and two lanes of the Scherk surface. Lines are added to the surface here for the sake of clarity.

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Fig. 7

According to a theorem of Richard Schoen, there exists no minimal surface that looks like a catenoid with a transverse tunnel in the neighborhood of the waist. As seen from above, the circular hole in the catenoid would be replaced by two holes with a handle between them. One can try to construct a surface in which one tries to push a tunnel from the two opposite sides (illustration at left). Each of the tunnel ends is perpendicular to a plane. If this attempt were successful, these two tunnel ends would move together and finally close, uniting the two tunnel ends, without a break, in a new minimal surface. However, the closer they approach, the greater the separation of the two vertcical parts of the original catenoid (illustration at right), and the attempt fails.

Triply periodic minimal surfaces

We turn now to a special class of minimal surfaces for which techniques for alteration are well developed, and which have generated great interest in neighboring sciences. We are speaking of surfaces, like the Gergonne surface illustrated in Fig. 2a, which make a certain kind of three-dimensional structure.

Such a structure—a so-called crystallographic cell—is so constructed that using it as a building block one can fill three-dimensional space with no gaps. By analogy with the situation in the plane, one speaks of the tiling of space. Besides the square prism (and a number of other shapes), suitable shapes for such a tiling include prisms whose base is either an equilateral triangle or a regular hexagon. Here we are not preferentially interested in reflection in edges, but rather in reflections in the traditional sense in suitable face planes of the building blocks, which have the same symmetry as the entire tiling.

Triply periodic minimal surfaces have—in the simplest cases, which we want to examine here—the same symmetry as that of a crystallographic tiling. In this context, interest lies almost exclusively in embedded surfaces. In particular these surfaces play a role in the description of molecular partitioning in crystals and in the separation surface between the

Stuttgart. They investigated the Coulomb field of point charges in a crystal lattice and discovered amazing similarities between periodic zero potential surfaces and the periodic minimal surfaces of Schwarz (Fig. 8*m*), Neovius (8*j*), and Alan Schoen (8*f*). In this connection, minimal surfaces play the role of prototypes of spatial structures. We see that as rule reality is more complicated than mere clear mathematical models.

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Fig. 8
One can join together crystallographic cells based on the equilateral triangle, the square, or the regular hexagon (colored tubes) in a gap-free tiling of space. The minimal surface modules in the cells satisfy corresponding symmetry conditions and fit together to make complete triply periodic minimal surfaces. Such surface modules can be combined with their neighbors to form tunnels in a variety of ways. They extend to the cell faces (*b*, *g*, and *m*), to the edge midpoints (*j*), to the corners of the crystallographic cell (*a* and *d*), or to a mixture of these elements (*e* and *f*). The H surface (*e*) and P surface (*m*) were found by Hermann Amandus Schwarz in 1865, and the S′-S″ surface (*f*), I-WP surface (*a*), and H′-T surface (*b*, *c*, and *h*) were found in 1970 by Alan Schoen. The complete Schwarz surface P shown in (*i*) consists of infinitely many specimens of the fundamental module inside cubes joined together at their faces. It separates space into two interpentrating labyrinthine regions. The red and green scaffoldings of tubes each lie in one of these labyrinths. The green tubes belong to the edges of the original cubes, while the red tubes connect the midpoints of adjacent cubes. Both of the tube scaffoldings and also both of the labyrinths are of the same shape. In contrast, the two labyrinths of the H′-T surface (*c* and *h*) are of different type, Thus, one can cut the complete surface into fundamental modules in two quite different ways. The trigonal labyrinth lies inside the triangular prism cell (*b*), while the hexagonal labyrinth lies inside the hexagonal prism cell. In fact, figures *b*, *c*, and *g* only show different pieces of one and the same minimal surface. The three surfaces*i*, *j*, and *k* belong together, because they all contain the same straight lines. The surface shown in *j* was found by Schwarz's student Erhard Rudolf Neovius, because in the group around Schwarz, the problem of minimal surfaces that contain the same straight lines was studied systematically. The surface *k* was constructed by one of the authors (Karcher) for this article. It contains, just as in the case of the Neovius surface, enough handles so that all of the symmetries—in particular the 180° rotations about the straight lines—are preserved.

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ecahedron can be built from six
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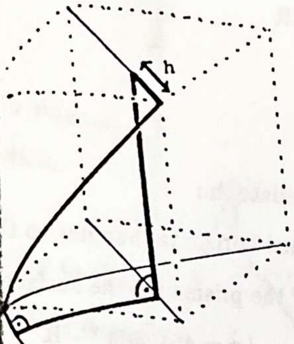
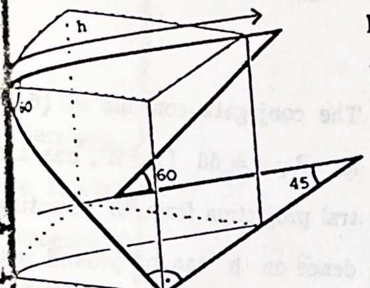


$h = 0$: F - Rd

$h = \frac{1}{2}\sqrt{2}$: P (zweimal)

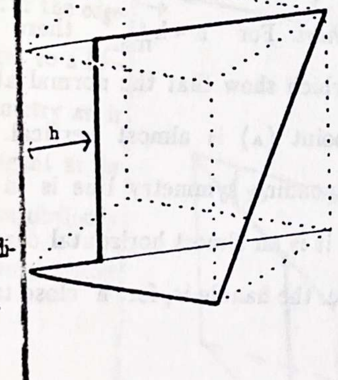
intermediate h : Surface with arms towards *all vertices of the* dodecahedron. This is the same as an *interior* P-handle glued to an F - Rd surface (punctured below 4-valent vertices).



$h = 0$: F - Rd

$h = 2 \cdot \sqrt{2}$: P (zweimal)

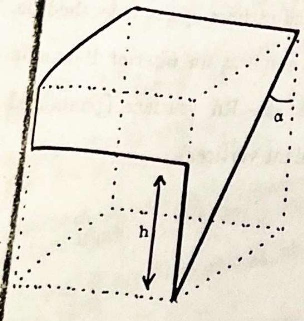
intermediate h : Surface with handles towards *all edges* (in C(P) fashion) of the dodecahedron.



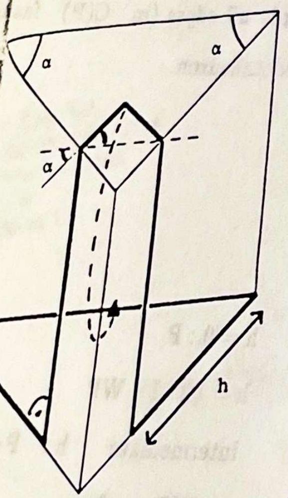
$h = 0$: P

$h = \sqrt{2}$: I - WP

intermediate h : P-handle glued into I - WP surface.



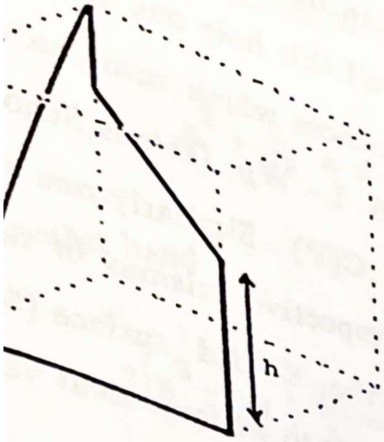
intermediate h :
additional horizontal handles to the vertical faces of the prisma for the surfaces (1.7)
S' - S" $(\alpha = 45)$, H" - R $(\alpha = 60)$,
T' - R' $(\alpha = 30)$.



The conjugate contour of (6.1.5), $\alpha = 45$ for P, $\alpha = 60$ for H, has a convex central projection from Z. Continuous dependence on h can be proved with Nitsche's arguments [N]. The limit $h \to 0$ gives a piece of the annulus on the P or H surface, i.e. for small h the handle is too *short*. For $h \to h_{max}$ there are barriers which show that the normal at the branch point (▲) is almost vertical. The corresponding symmetry line is in the vertical *ymmetry plane through the handle; it is an almost horizontal curve because of* *he control of the normal along it, i.e. the handle is, for h close to h_{max}, too* *ong.*

.3. Conjugate contours for the examples in (6.1) with inward handles or related to the rhombic dodecahedron.

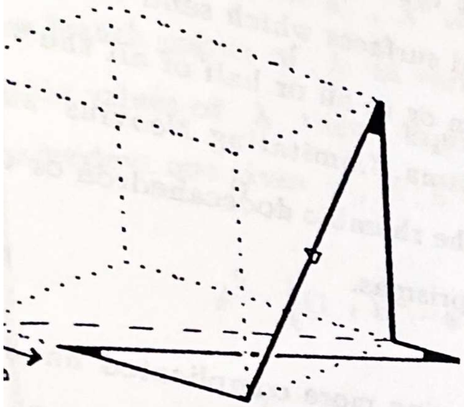
The rhombic dodecahedron can be obtained by reflecting the midpoint of a cube



h = 0 : I - Wp (5.3.3)

h = 1 : P-surface

intermediate h : Schoen's O, C - TO (6.1.2)



h = 0 : I - Wp

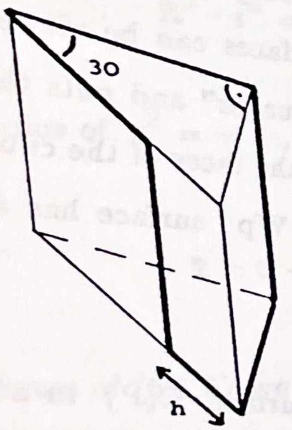
h = 1 : C(P)

intermediate h : symmetry lines of I - Wp and C(P) on faces of a cube.

$\int K dO = -2\pi \cdot \frac{5}{8}$

$3 \times \frac{1}{8}$ double branch point: $-\frac{3}{8}$

$1 \times \frac{1}{2}$ simple branch point: $-\frac{1}{4}$



h = 0: T' - R' (1.7)

$h = h_{max}$: H'' - R (1.7)

intermediate h: S' - S'' in the prisma of (4) ustment.

that one should be prepared to find
nimal surfaces in $\mathbb{R}^3$. I do not see
o look for a classification. I found it
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e the method explained in section 4: Th
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h: - S" (1.7) in a brick rather than a qua-
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h, in general, parallel planes actually
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lles" (see 2.5.2). For example the sur-
re many symmetry lines meet vertical-
cubical cell. One can puncture the mi-

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al surface there and make a catenoid like hole with its bounding symmetry
parallel to the face of the cube. Pull this hole out until its symmetry plane
n the face of the cube. We get surfaces which combine on the face of the
e the symmetry lines of (i) P and I - Wp (this is Schoen's 0 , C - TO),
) P and C(P) , (iii) I - Wp and C(P) . Similarly one can pull additional
dles to the vertical faces of the respective prismas in the cases S' - S" ,
- R , T' - R' (1.7). In the case of the F - Rd surface (5.3.2) one can pull
ditional handles either to all trivalent or to all fourvalent vertices.

.1.3. Another option is to adapt the forms we have seen in section 5 to other
lls. Immitating I - Wp we find surfaces which send their arms to all the *ver-*
ces of the rhombic dodecahedron or to all or half of all the vertices of a trian-
lar, quadratic or hexagonal prisma. Immitating Neovius' surface C(P) one
n have handles to all *edges* of the rhombic dodecahedron or to all the horizon-
l edges of the three orthogonal prismas.

.1.4. There is the possibility of using more complicated handles. For example
ne can pull the six punctures of I - Wp or C(P) of (6.1.2) *inward* and have
hem join up in the middle - as if we had glued in a handle like the P-surface's
ndamental piece. (Some previous surfaces can be viewed in this way: If one
ts the I - Wp surface along its "equator" and puts the bottom half on the
op then this fundemental piece meets the faces of the cube in the same pattern
s the S' - S" surface; only the I - Wp surface has an additional interior
ross-handle.)

.1.5. Schoen's surface C(H) or the surface C(P) in a quadratic prisma can
e obtained from the "triangular" resp. "quadrilateral" catenoids (5.2, 5.1.3) by
ulling 3 resp. 4 horizontal handles to the vertical edges of the surrounding

$\pm r$, $\pm iR$ the values of λ where $g^4 = -1$, then

$$g^2 + \frac{1}{g^2} = C \cdot (\lambda^2 - r^2) \cdot (\lambda^2 + R^2) \cdot (\lambda^3 + \lambda)^{-1} .$$

he other hand

$$2\left(g^2 - \frac{1}{g^2}\right)\frac{dg}{g} = C \cdot \frac{(\lambda^2-a^2)\cdot(\lambda^2+b^2)\cdot(\lambda^2-\rho^2)}{\lambda\cdot(\lambda^2+1)^2} \cdot \frac{d\lambda}{\lambda} ,$$

re we obtained the denominator by differentiating the previous formula, the erator since $\lambda^2 = a^2$, $\lambda^2 = -b^2$ give branch points of g and $g^4 = 1$ es branch points of λ at vertices of the pentagon but not on the edge: $\pm \rho$ the values of λ there. Equating the last expression with the derivative of previous one gives ρ, R, r in terms of a, b:

$$\rho^2 \cdot \left(1 + b^2 - a^2 + 3a^2b^2\right) = -3 - a^2 + b^2 - a^2b^2$$

$$R^2 - r^2 = 3 + \rho^2 + a^2 - b^2$$

$$R^2 \cdot r^2 = \rho^2 a^2 b^2 .$$

sertion of $g^2 = -1$, $\lambda = +\rho$ into the equation finally determines C

$$-2 = C \cdot \left(\rho^4 + (R^2 - r^2)\,\rho^2 - r^2R^2\right) \cdot (\rho^3 + \rho)^{-1} .$$

Because of (5.5.5) we know that this 2-parameter family of minimal surfaces ontains a 1-parameter family of figure eight bands.

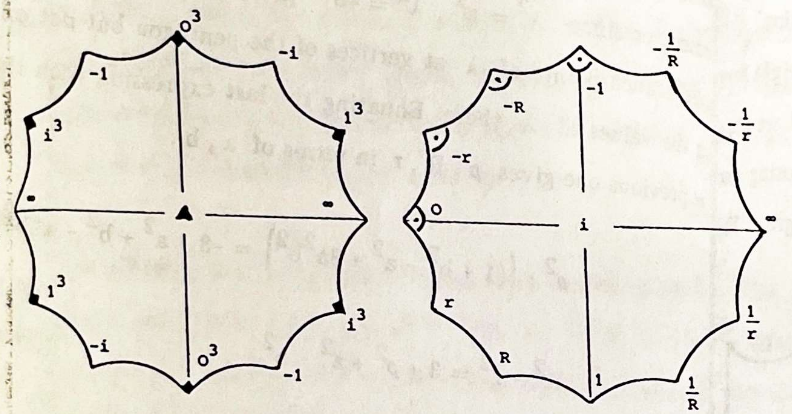
Consider two great circle arcs
a , b of length $\frac{\pi}{2}$ on one of
the Clifford tori which meet
the two axis $s = 0$, $s = \frac{\pi}{2}$
at points distance $\frac{\pi}{k}$ apart.
Their endpoints can be
joined by other great circle
arcs c , d of length $\frac{\pi}{2}$
which lie on the two other
- Clifford tori which meet the
cs form a quadrilateral a , c , b , d .
L], to produce his embedded minimal
$+ \frac{\pi}{N}$ (recall m = n = 1) to get an-
The two quadrilaterals bound an an-
tori. If N is large we can find a tube
tor pointing inward. Finally we mini-

n extend this Plateau ring to a closed
d symmetry group maps the union of
s possible to prove that one such Pla-
copies. This implies embeddedness of

l surfaces in S^3 are not the theme of
e could expect that the constructions
S^3 .

Weierstraß representation for the Neovius surface C(P) and the figure eight annulus.

1. For the C(P) surface (5.5.1) we have degree(g) = 8 , degree(μ) = 24 .
s is so large that I will only treat the case with maximal symmetry. The six
damental quadrilaterals with one vertex at one of the eight points on a space
gonal of the cube fit together to a 90°-12-gon whose edges are horizontal
d vertical planar symmetry lines. An auxiliary function λ of degree 4 can be
ined by mapping this 12-gon to a hemisphere. The straight lines on this sur-
e are diagonals of the quadrilaterals. The 12-gon therefore is in the hyperbo-
picture equilateral and this implies the values of λ at all vertices of the
gon. The data are



g , degree 8

λ , degree 4
real on the boundary,
midpoint normalized to i ,
$r = \text{tg } 15° = 2 - \sqrt{3}$, $R = \text{tg } 30° = \frac{1}{3}\sqrt{3}$

$$\mu \approx \left(\lambda - \frac{1}{\lambda}\right)^2 \cdot \left(\lambda^2 + \frac{1}{\lambda^2} - r^2 - \frac{1}{r^2}\right)^{-1} \cdot \left(\lambda + \frac{1}{\lambda}\right)^{-1} ,$$

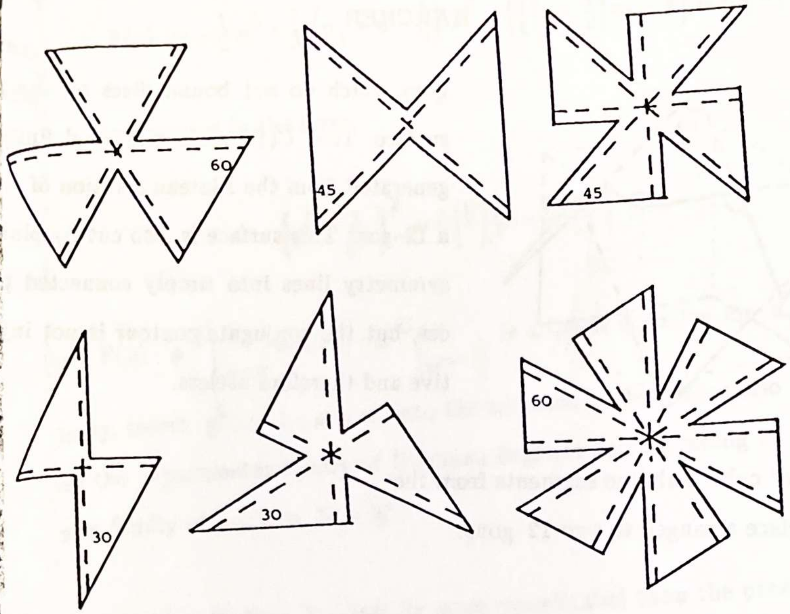
nce this function is real on the boundary and has the correct zeros and poles.
inally (with $r = \text{tg } 15°$, $R = \text{tg } 30°$):

s the annuli treated in (5.1.
H , RII , RIII). Consider a
ontours which converge to a
gure eight" as in the sketch
one a small distance vertically
wish to solve the annular Pla
n also for the limiting case.
following barrier: Take two
h that the left square (and the
of the figure eight surrounds
the catenoid C_2 . We need
es of the bottom eight and the
tenoid pieces by a rectangular

of annular Plateau problems -
on, also the limiting annular

edges extends the annulus to

ngular tessellations of (5.2) -
ys to the (45,45,90)-triangles
sketches also show approxi-



5.6. A variation of (5.5.5).

one replaces the lower figure eight by a square vertically below the vertices of e top eight then one has a contour into which one can span an annulus with e same barrier as before. It extends to a triply periodic minimal surface.

5.7. Annuli-constructions in S^3 .

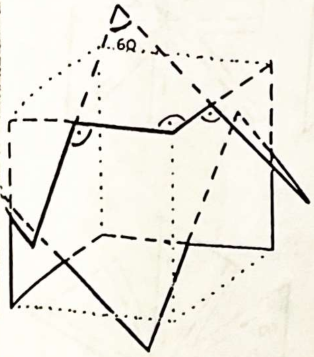
When J. Pitts visited Bonn in summer 1987 he told me that he and Rubinstein here using their sweep-out method to construct minimal surfaces in S^3 by tarting with k spherical helicoids (1.6.2) which had the two axis $s = 0$ and $= \frac{\pi}{2}$ in common and along them intersected with equal angles $\frac{\pi}{k}$. (To avoid onorientable surfaces take m , n in (1.6.2) not even, and relatively prime to void unnecessary coverings.) Pitts and I then discussed the possibility of also getting such surfaces by explicit - annular - Plateau problems. I describe the esult for the case $m = n = 1$ (Clifford tori). The figure indicates tereographic projection of S^3 .

.ce C(P) has 12 arms to *all* edges
ut this surface into (45,60,45,90)-
rals; the normals at the vertices
hree edge directions of the cube
space diagonal. From this we ob-
onjugate contour, hence existence
again, and constant mean curva-
panions with (2.3).

ay as in (5.5.1) the same lines as

on the D-surface consist of the
le tessellation cubes and further
ecting edge-midpoints; these en-
very second cube, say the black
black and white coloured tessel-

s on the D-surface are arranged
oundaries of this annulus are 12-



gons which do not bound discs on the D-surface. The C(D) surface is by definition generated from the Plateau solution of such a 12-gon! This surface is also cut by planar symmetry lines into simply connected pieces, but the conjugate contour is not injective and therefore useless.

hite" cube with line segments from the
surface arranged to two 12-gons.

5.3. C(H) denotes a surface which has the same lines as the H-surface (5.2),
t I can prove its existence only via a 1-parameter problem (6.1.5).

5.4. The idea explained in (5.5.1, 5.5.2) can also be used in S^3. In [KPS] we
ed the selfdual tessellations of S^3 by five 120°-tetrahedra resp. by 24 120°-
tahedra to find minimal surfaces meeting the faces of the tessellation in the
me way as the P-surface. These surfaces were constructed by Plateau solu-
ons for quadrilateral contours, since there are many great circles in S^3 such
at 180°-rotation around them maps the tessellation onto the dual one and the
inimal surface to itself (so that these great circles are on the minimal surface).
hree such quadrilaterals link to an annulus on the minimal surface. The
lateau solution of one of its bounding hexagons (angles $\frac{\pi}{2}, \frac{\pi}{3}, \ldots$, resp.
$, \frac{\pi}{4}, \ldots$) generates another embedded minimal surface. The one from the te-
ahedral tessellation has 20 such hexagons and $\chi = -10$, the other one has 192
exagons and $\chi = -24 \cdot 7$.

$$P(\mu) := -\frac{1}{2}\mu^3 + \frac{1}{3}\left(u_1 + u - \frac{1}{u}\right)\mu^2 - \frac{1}{4}\left(\left(u - \frac{1}{u}\right)u_1 - 1\right)\mu + \frac{1}{5}u_1$$

get the equation in the form:

$$\left(g^3 - \frac{1}{g^3}\right)^2 = \tilde{C}\left(\frac{P(\mu)}{\mu^5} - \frac{P(u_1)}{u_1^5}\right) .$$

Next, $P(\mu) \cdot \mu^{-5}\Big|_{\mu=u} = P(\mu) \cdot \mu^{-5}\Big|_{\mu=-\frac{1}{u}}$ is a linear equation for $u_1 = u_1(u)$. Finally, insert $g^6 = -1$, $\mu = u$ into the equation to determine also $\tilde{C}(u)$ - and thus the 1-parameter family of Riemann surfaces corresponding to the 1-parameter family of surfaces T' - R'.

5.4.7. The last surface, F - Rd , is more complicated than the previous one and not worked out.

5.5. Surfaces which contain exactly the same lines as earlier ones, including Schoen's C(P) , C(D) .

Schwarz and in particular his student Neovius were very interested in arrangements of lines which could be shown to bound different minimal surfaces. C(P) was discovered by Neovius [Ne].

5.5.1. From line segments (= hexagondiagonals) on the most symmetric Schwarz surface one can make a (90,60,90,60,90,60,90,60)-octogon which is homotopic on the surface to one of the convex symmetry lines on the face of the cube. (Orthogonal projection of the octogon to that face is a square.) The Plateau solution for this octagon is definitely not on the P-surface. It generates a complete minimal surface carrying the same lines as the P surface. (Therefore Schoen's name C(P).) In the same way as S' - S" (1.7) has four arms to the

R

$$\cdot\left(\lambda-\frac{1}{\lambda}-c+\frac{1}{c}\right)\cdot\frac{d\lambda}{\lambda}$$

e C(F

it this

rals; t second factor of $\frac{d\lambda}{\lambda}$ vanishes, i.e.

hree e

pace

$g^3-\frac{1}{g^3}\Big)=0$,

onjuga

agair

tion. We take c as independent

panior

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$\Big)^{-1}\cdot\left(\lambda+\frac{1}{\lambda}\right)\left(\lambda-\frac{1}{\lambda}-A\right)$.

ray

of the form

on

ne $\frac{+b}{\mu^3}+\frac{1}{2}\frac{ab}{\mu^2}$;

ect

ver nstants have to be chosen solving

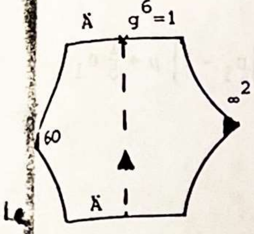
bl

$\big)=a^4\left(\frac{2}{r}-1\right)$.

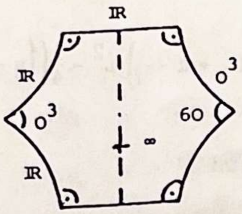
the previous one. By reflection in

t we again obtain hexagons which

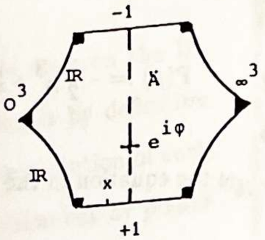
y function λ. The data are



g , degree 4



μ , degree 12



λ , degree 6

real on the boundary,

No further branch points

normalization:

equator = symmetry

line

Let a be a branch value of g , then start with

$$g^6+\frac{1}{g^6}-a^6-\frac{1}{a^6}=-\frac{1}{2}\mu^{-2}+\frac{1}{3}(r+R)\,\mu^{-3}-\frac{1}{4}rR\,\mu^{-4},$$

i.e.

$$6\left(g^6-\frac{1}{g^6}\right)\frac{dg}{g}=(\mu-r)(\mu-R)\cdot\frac{d\mu}{\mu^5}\ .$$

The constants $B:=a^6+\frac{1}{a^6}$, r , R are determined from the ratio $v:=\frac{R}{r}$ via

$$144\cdot\frac{B-2}{B+2}=v^3\,\frac{2-v}{2v-1}\ ,\quad 12(B-2)r^2=2-v\ .$$

Or using λ :

$$\mu\approx\left(\lambda+\frac{1}{\lambda}-2\cos\varphi\right)^{-1}$$

$$3\left(g^3+\frac{1}{g^3}\right)\frac{dg}{g}\approx\left(\lambda+\frac{1}{\lambda}-2\cos\varphi\right)\cdot\left(\lambda+\frac{1}{\lambda}-x-\frac{1}{x}\right)\frac{d\lambda}{\lambda}\ ,$$

$$i\left(g^3-\frac{1}{g^3}\right)=\mathcal{C}\cdot\left(\frac{1}{2}\left(\lambda^2-\frac{1}{\lambda^2}\right)-\left(\lambda-\frac{1}{\lambda}\right)\cdot\left(x+\frac{1}{x}+2\cos\varphi\right)\right)$$

i.e. $\cos\varphi=-\left(x+\frac{1}{x}\right)^{-1}$. Determine $\mathcal{C}$ from $g^3=i$, $\lambda=x$.

before immediately

$$\mu = \lambda^2 \cdot (\lambda^2 - 1)^{-1} ,$$

d, since the branch points $\neq 0$ of λ are precisely at $g^4 = 1$ and the branch ints of g $(\neq 0 , \infty)$ are where $\lambda^2 = 1$ we have

$$g^2 + \frac{1}{g^2} = \frac{1}{2}\left(a^2 + \frac{1}{a^2}\right) \cdot \frac{3\lambda^2 - 1}{\lambda^3} ,$$

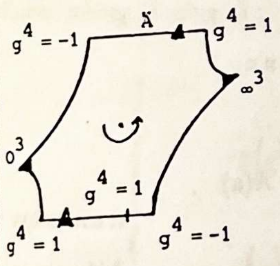
ance

$$2\left(g^2 - \frac{1}{g^2}\right)\frac{dg}{g} = \text{const} \cdot \left(-\frac{1}{\lambda} + \frac{1}{\lambda^3}\right)\frac{d\lambda}{\lambda} .$$

gain, with more effort, one finds with $A := a^4 + a^{-4}$ and a different normalition for μ a second representation:

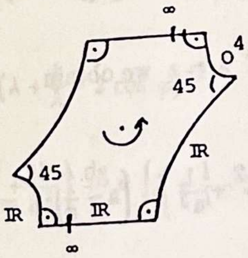
$$g^4 + \frac{1}{g^4} - A = -\frac{\mu - r}{\mu^3} , \quad 2 + A = \frac{4}{27} \cdot r^{-2} .$$

4.3. For the I-Wp surface (5.3.3), there is also a translational identification etween the middle plane and the top plane. This makes it more difficult to nd further functions, but halfing the degrees of g and μ is also a good simification. We collect the data for g and μ on the centrally symmetric tessalting hexagons (3.5).

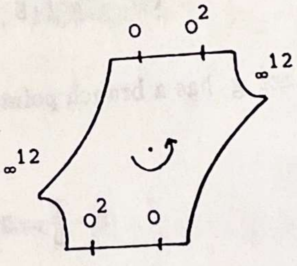


g , degree 3

No other branch points!



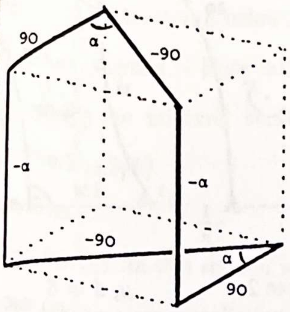
μ , degree 8



$g^4 + \frac{1}{g^4} - 2A$

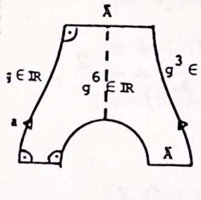
s in quadratic prismas and also for an analogous one which looks at every nd vertex of a hexagonal prisma, see figures.

faces and symmetry planes of the prisma cut the expected surfaces into 16 p. 12 hexagons with angles $(\alpha,90,90,\alpha,90,90)$ and normal rotations 0,α,90,-90,α,90) - where $\alpha = 45°$ for I - Wp and $\alpha = 60°$ for the other (which I denote analogously: T - Wp , Wp for wrapped package). - Indeed we find a conjugate contour on the surface of a brick with all required angle conditions and one 180°-rotational symmetry. Therefore the two surfaces and theirconstant mean curvature companions exist. (The spherical polygon (2.3) also has the 180°-symmetry.)



Weierstraß representations for the surfaces in (5.3).

.1. Following (3.5) we write for the case H' - T (5.3.1) the knowledge about e functions g and μ at the hyperbolic hexagon obtained by reflecting the ntagon of (5.3.1) at the vertical diagonal symmetry plane. We guess another nction λ on the Riemann surface by mapping the hexagon to a half sphere.

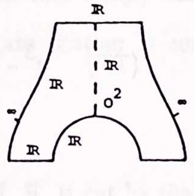


g , degree 3

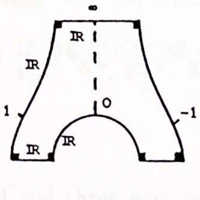
No further branch points,

other branch values

$\xi \cdot a$, $\xi \cdot \frac{1}{a}$, $\xi^6 = 1$



μ , degree 12



λ , degree 6

λ normalized to

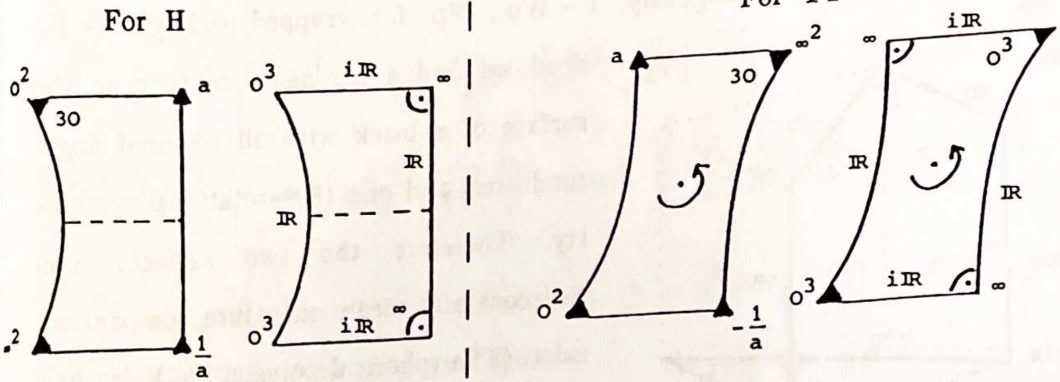
±1 at the branch

points of g .

the edge therefore gives a
ntagon, bounded by planar
angles (90,90,90,90,60) and
(0,60,-90,90,-60). This gives
nalogous to (1.7.3). Its Pla-
onjugation again gives the
au piece itself also generates
nly immersed surface.) Con-
e neighbours of Schwarz'

a similar way have rather
made out of "twisted tri-
ar Plateau problem we ro-
idpoint by 180°. I call the
lanar symmetries, but the
lus into six quadrilaterals
adjacent copies of the an-
ymmetries of the complete
. or the conjugate Plateau
uli, the unstable ones are

.4. Following (3.5) our knowledge about the functions g and μ on the qua-
laterals is as follows:


For H

For TT

g, degree 2 μ, d = 6

g, degree 2 μ, d = 6

o other branch points in the quadri-
terals, other branch values
a, $\rho \frac{1}{a}$ with $\rho^3 = 1$

No other branch point in the quadrilaterals, other branch values
ρa, $-\rho \cdot \frac{1}{a}$ with $\rho^3 = 1$.

he values of g on the boundary are on the meridian IR and $\bar{\rho} \cdot$ IR . Since the ranch values of g are not assumed elsewhere one obtains the equation imme- iately (use (3.5.2) to normalize the function μ, making its coefficient $= \pm 1$):

H) $$g^3 + \frac{1}{g^3} - a^3 - \frac{1}{a^3} = -\mu^2 ,$$ (TT) $$g^3 - \frac{1}{g^3} - a^3 + \frac{1}{a^3} = +\mu^{-2} .$$

.3. Further surfaces as in (1.7), including Schoen's surfaces H' - T , S' - S" , H" - R , T' - R' , I - WP , F - Rd.

describe the surfaces which are not yet treated in (1.7) and give the conjugate ontours.

.3.1. The surface H' - T meets all faces of an orthogonal prisma over a regu

2. Surfaces obtained from annular Plateau problems, including H, RII, RIII.

emark. Recently [KF2] was sent to me. By working through all the space roups the authors classify all the annuli which are bounded by convex olygons in parallel planes and extend to embedded triply periodic minimal urfaces. In addition to the annuli which I describe below there is another less bvious type: Consider a rectangular net whose diagonals meet under 60°. The ets above and below are obtained by ±120°-rotation plus a translation by half diagonal. - Note that the determination of these families of lines does not give ll the minimal surfaces which carry precisely these lines, for examples see 5.5.5/6).

.2.1. In this section we extend the construction of (5.1.3) to some other annuli. There are tessellations of the plane by rectangles, equilateral triangles and two 0°-triangles (45, 45 and 30, 60) such that 180°-rotations around all the edges reserve the tessellation. Translate this tessellation vertically up to a second lane by a sufficiently small amount such that the annular Plateau problem for pair of triangles vertically above each other can be solved, to give "triangular atenoids". These can be completed by rotation around edges to complete embedded surfaces which Schoen called $H = RI$, RII, RIII. RIII is not cut by planar symmetry lines into simply connected pieces. RII has such symmetry ines but the conjugate contour is complicated, in particular not a Nitsche graph (4.2).

.2.2. One annulus of H is cut by the horizontal and three vertical symmetry planes into 12 quadrilaterals (90,90,90,30) which are bounded by one straight egment and three planar lines of symmetry.

t difference of their endpoints is positive resp. negative. The intermediate e theorem now gives existence of the surface described in (4.1).

Examples of minimal and constant mean curvature surfaces in $\mathbb{R}^3$ without parameter adjustment. Minimal surfaces in S^3.

ly periodic embedded minimal surfaces decompose $\mathbb{R}^3$ into two compo- ts (Schoen: labyrinths). Schoen imagined his surfaces as common boundary ch labyrinths and he described these in terms of pairs of "skelettal graphs", piecewise linear 1-dimensional deformation retracts of the labyrinths. His es are derived from this description, but I find it difficult to explain them ore detail. I will mention the names for reference reasons without further ments.

is section covers all but two of Schoen's examples: P , D , G , CLP , H , P) ; RII , RIII , H' - T ; S - S' , H" - R , T' - R' , I - WP ; C(D) , F - Rd ; o others in the same spirit and seven different ones. The first five (not count- G) were known to Schwarz and his school [Sc,Ne]. The existence proofs her on the basis of section 1 or with the help of annular Plateau problems are y. If we can use section 1 then deformation into constant mean curvature rfaces (2) is possible. Some emphasis will be on Weierstraß representations. (5.5.4) and (5.5.7) I construct new minimal surfaces in S^3 . - Examples in requiring parameter adjustment (4) will be discussed in section 6.

1. The P , D , G and CLP surfaces.

1.1. The P-surfaces were treated in (1.6.2). Their conjugates - called D-sur- ces - are also triply periodic but usually not cut by planar symmetry lines in- simply connected pieces. The most symmetric P-surface carries many addi-

. Remark. Since we know (in the later examples) the zeros and poles of g
μ and the branch points and branch values of g there is a number of
s where the equation between g and μ is fairly obvious. In other cases the
tion μ is rather complicated and with luck one finds a simpler function.
example we can define a function λ on the Schwarz surface by mapping
emann!) one hexagon to a hemisphere bounded by the real line and for nor-
ization map the branch point of g to (the midpoint) i, then we extend the
inition to the Riemann surface by reflection. The degree of λ is 4, that of
s 8.

.4. Another use of the symmetry lines.

r any curve γ along which the Gauß map has *values* on a fixed meridian or
equator (compare 3.2) we have

$$\left(\frac{dg}{g}(\dot{\gamma})\right)^2 \in \mathbb{R} .$$

ce we represent $dh = \mu \cdot \frac{dg}{g}$ it follows now from (3.2.1):
On horizontal or vertical straight lines we have $\mu \in i\mathbb{R}$,
on horizontal or vertical planar symmetry lines we have $\mu \in \mathbb{R}$.
is is additional specific information beyond the zeros and poles of μ.

. Global Weierstraß representations of Schwarz surfaces (1.6.2).

e write our knowledge of the functions g, μ (3.5.2) and λ (3.5.3) at the hy-
rbolic hexagons. Abbreviate degree = d, equator = $\tilde{A}$, mark known branch
ints by ▲ or ▬. ($\mu = 1$ where $g = 1$ is a normalization.)

ametrized by Riemann surfaces of genus 3 . The catenoid and the (con-
helicoid are parametrized by $S^2 \setminus \{0,\infty\}$.

he hyperbolic picture.

In terms of hyperbolic geometry a much more precise description of the
ying Riemann surface can be obtained from the symmetries of the mini-
rface. Since isometries of the minimal surface are conformal automor-
s they are also isometries for the hyperbolic metric on the Riemann sur-
The symmetry lines are - as fixed point sets of isometries - also hyperbo-
odesics. The geodesically bounded pieces M^2_* are then hyperbolic poly-
For example the Schwarz Riemann surfaces (1.6.2) are tessellated by
congruent, centrally symmetric, 90°, hyperbolic hexagons. Such hexagons
d on only two edgelength parameters, therefore we see a two-parameter
y of Riemann surfaces - as we expected since up to scaling the brick de-
s only on two parameters.

Summarizing we now know the range of the Gauß map in terms of spheri-
geometry and the domain in terms of hyperbolic geometry, i.e. the Gauß
is (via the Riemann mapping theorem) determined by just looking at the
ce. The same is true for the differential dh . Obviously we have to ask:
this be made more explicit?

Equations for the Riemann surface.

section and its applications are joint work with Meinhard Wohlgemuth

1. From a complex analytic point of view a compact Riemann surface is also
by two meromorphic functions which separate points. The function field

mean curvature is zero if and only if the linear functions on $\mathbb{R}^3$ restricted the surface are (Laplace Beltrami) harmonic. Locally we define the conjugate of a harmonic function u by grad $u_* := D^{90°} \cdot \text{grad } u$ and observe that $= u + iu_*$ is holomorphic. If h is obtained from the height function and g the Gauß map composed with orientation reversing stereographic projection the standard Weierstraß representation [O] is:

1.1. $$F := \text{Re} \int \left(\frac{1}{g} - g \,,\, i\left(\frac{1}{g} + g\right) ,\, 2\right) dh$$

1.2. $$ds^2 = \left(|g| + \frac{1}{|g|}\right)^2 \cdot |dh|^2$$

1.3. $$\langle S{\cdot}w,w\rangle_z = 2 \cdot \text{Re}\left(\frac{dg}{g}(w) \cdot dh(w)\right) \;,\quad w \in T_zM^2 \;.$$

bserve that 3.1.1 is in fact a global representation: the Gauß map is globally fined and although h may not be globally defined, the differential dh is ven in terms of the global height function and the metric 90°-rotation.

.1.4. Usually one calls the imaginary part of (3.1.1) the conjugate minimal surce F_*, compare (1.1).

.2. Symmetries.

one reconstructs a minimal surface from its Weierstraß representation one ants to recognize its symmetries as quickly as possible. Since the connection etween the function g and the coordinate free Gauß map is via a stereograhic projection which distinguishes the vertical direction, we do not completely cceed. We find a simple criterion for horizontal or vertical symmetry lines along which the Gauß map has values on the equator or on a fixed meridian).

the boundary arcs of co
for minimal surfaces (L
arc which starts at a vert
other arc at v *stays* i
worse - see the immersed
the constant mean curv
on of the spherical conj
oen's minimal surfaces cu
more specific statements
in (2.4).

ean curvature 1 . Let L_i
.,n-3) . For sufficiently sm
ately $\epsilon \cdot L_i$ (i = n-2,n-1,
$\frac{1}{\epsilon}$, we obtain surfaces of co
ginal minimal surface. The
rfaces - for small ϵ - c
surface. Also, they are e

varz surfaces (1.6.2).
wo adjacent vertices span
ring edges. For sufficiently
spherical brick by intersec
erical polygon Q . The Pla
nd this removes the 2π-un
f fields are involved, there
ly and also discuss Q fo
n be done without compu

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n: The additional symmetry lines (which connect the midpoint of a hexa-
to the midpoints of its edges) cut the hexagon into six quadrilaterals. Each
ese has one edge which gives a symmetry arc on the face of the cubical cell.
ing this edgelength go to zero we expect the constant mean curvature sur-
to converge to spheres which are inscribed into the cubes of the tessella-
. The six quadrilaterals in a hexagon converge to (60,90,45)-triangles which
ellate a (90,90,90)-triangle. *From this degenerate situation we can easily*
the nearby quadrilaterals by using the knowledge of (2.2., 2.3): Choose one
ngle edge to the 45o-vertex, right rotate it along the edge to its other end-
nt (staying in the Hopf fibration determined by the chosen edge), finally,
se the gap which opened at the 45o-vertex by a common perpendicular
hich is on the Hopf fibration determined by the triangle normal at the
-vertex)! Quadrilaterals are obviously on the boundary of their convex hull
his eliminates the 2π-uncertainty in the rotation of the surface normals. We
re found the contours in S^3 which give constant mean curvature surfaces in
which meet the faces of a cube in very small convex curves; the surface
ks like a cubical packing of spheres but with slightly too small spheres and
en connected with Delauney waists. - Instead of "closing the gap" with the
ry short common perpendicular we can also choose the long ($\approx \frac{\pi}{2}$) perpendi-
ar which is $\frac{\pi}{2}$ away from the short one (on each edge). This "large" quadri-
eral is still on the boundary of its convex hull so we control the rotation
gles completely. The surface in $\mathbb{R}^3$ is now made of two types of building
cs: Spheres with three opposite pairs of Delauney waists and between them
heres with two opposite Delauney waists (the equators of all the spheres are
mmetry lines, the waist-lines are not).

5. Connection with Lawson grids.

e extend the last examples to a less symmetric situation: Put the six quadri-

– length (β) .

. "support surface" along

tuation: If a , b , c were

(β) – is a trivial observatio

(b,c) rotate because of (2.1.

locity -1.

constructive corollary. It im

be computed explicitly vertex

om the minimal surface pieces

piece $M^2_+ \subset \mathbb{R}^3$. The total rot

e edge corresponding to β is

), span (b,c)) .

tours.

g) that we have constructed as

by conjugating the Plateau sol

sh to deform the minimal surface

2, 1.4) and have to find a spher

try planes of the constant mean

will always be used, it is not

---

1. Assume further (for the simplicity of the description) that P has an edge hich meets its adjacent edges γ_1 , γ_2 *perpendicularly* and that γ_1 , γ_2 are *parallel.* We call β the *last edge* of P .

choose a vertex v of P not adjacent to β as starting vertex, orient P parallel translate all the tangent vectors of the edges to v . We obtain a ction of vectors

$$a_1,\dots,a_{n-3},b,c_1,c_2 \ .$$

2. Consider these data in some tangent space of S^3 and define the *right* ro- g Hopf vectorfields determined by $a_1,\dots,a_{n-3},b,c_1,c_2$. Assign arbitrary sufficiently small) lengths $\ell_1,\dots,\ell_{n-3}$ to the a-edges and connect – in the e order as on the Euclidean contour P – great circle arcs of length ℓ_i in direction a_i to obtain a piecewise great circle arc $\tilde{Q}$. Then take great cir- γ_1 , γ_2 through the endpoints of $\tilde{Q}$ in the directions c_1 , c_2 . Since c_1 , are linearly independent there are for small ℓ_i exactly two great circles , β_2 orthogonal to γ_1 , γ_2 and meeting each γ_i at four points $\frac{\pi}{2}$ apart. consider the β_i as integral curves of *one* right rotating Hopf field; this has be ±b since there is no other field perpendicular to c_1 , c_2 . Orient the β_i the direction +b . Finally we have two choices among the oriented arcs on β_i to connect γ_1 , γ_2 in such a way that the edgelengths on γ_i are $< \pi$ the edges of this closed polygon Q are oriented in the same way as on P . apply (2.2.2) to every edge of Q and find that the normals of the constant an curvature surface (made via (1.2) from a Plateau disk bounded by Q) n along the planar boundary arcs through (mod 2π) the same angles as on minimal surface (made by conjugating the Plateau disk spanning P). This mpletes the construction of the spherical contour Q .

jection. M^2 is a geodesic
along the edges by 90°,
e six boundary arcs of the
ogonal pairs of parallel
rick B_* orthogonally.
M^2 is a spherical 90°-tria
is on the surface M^2 and
through m is an isometry
one simple branch point in th
.2) we now have: the bounda
turn monotonely through 90
of the branch point normal
oundary, i.e. M^2_* is a graph
symmetry group which exten
n $\mathbb{R}^3$. - The cubical version wa

hows that one should in genera
ction. The embeddedness proo
dary polygon and is rather inde-
ill not come back to the embed-
l point out that the "obvious"
convexity properties of the boun-

R'.
s of this paper before I come to
ree more examples now for two

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ons: To apply the conjugate surface construction to examples not known in last century and to supply the "simpler examples" which are helpful to ex- the 1-parameter method mentioned in the introduction.

.1. We consider $\mathbb{R}^3$ tessellated by orthogonal prismas over (i) squares choen denoted this case S' - S"), (ii) regular hexagons (H" - R), (iii) regular angles (T' - R').

flections in the faces and the symmetry planes of these prismas generate the mmetry groups of the minimal surfaces to be constructed. The minimal surfa- s are to meet the top and bottom face of the prisma in convex curves (as hwarz surface), but the horizontal connection to the neighbouring prismas is ross the vertical edges (not the faces as in 1.6.2), see figures. From this quali- tive description we expect pentagonal fundamental pieces whose bounding mmetry lines have the following properties:

7.2. The horizontal symmetry line on the top face rotates through (i) 45°, (ii) 0°, (iii) 60° and meets two vertical symmetry lines orthogonally. Along one ertical symmetry line the total turn of the normal is 0° and it meets the ther horizontal symmetry line orthogonally. This other horizontal symmetry ne turns -90° and then meets the third vertical symmetry line (the one on a risma face) orthogonally.

The second and the third vertical symmetry line turn -90° resp. +90° and eet under (i) 45°, (ii) 60°, (iii) 30° on a vertical edge of the prisma.

This is enough to construct a polygonal conjugate contour P which in fact oes have all the expected properties. The Plateau solution of P therefore has conjugate surface whose symmetry lines indeed behave as expected:

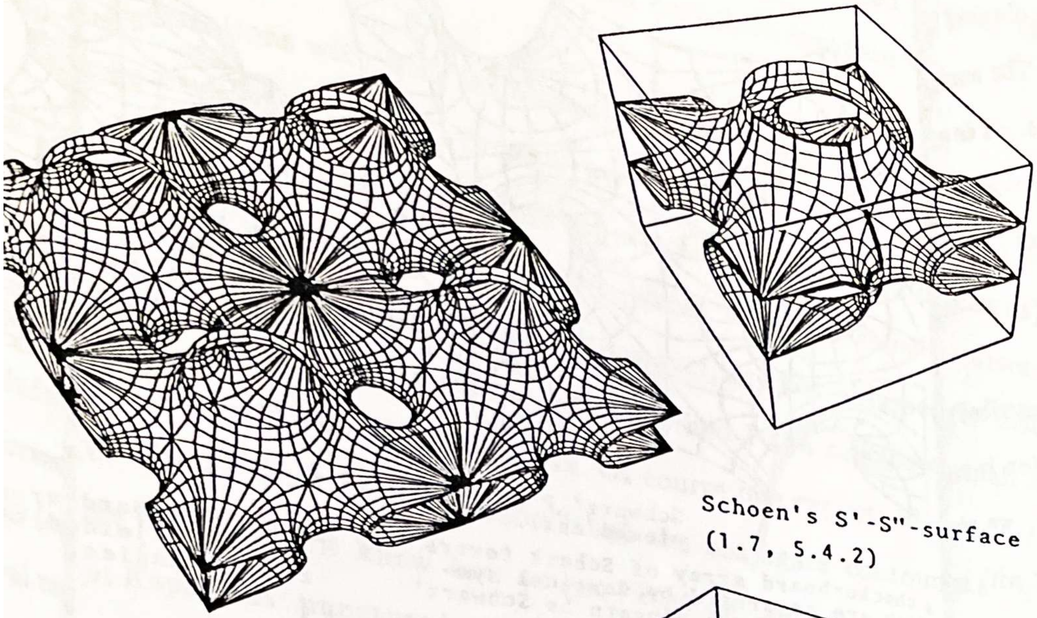
^{2} then $\kappa = 0 = \tau_*$, i.e. the conjugate surface M^2_* is bounded by planar odesic arcs.

3.2. Reflection principle.

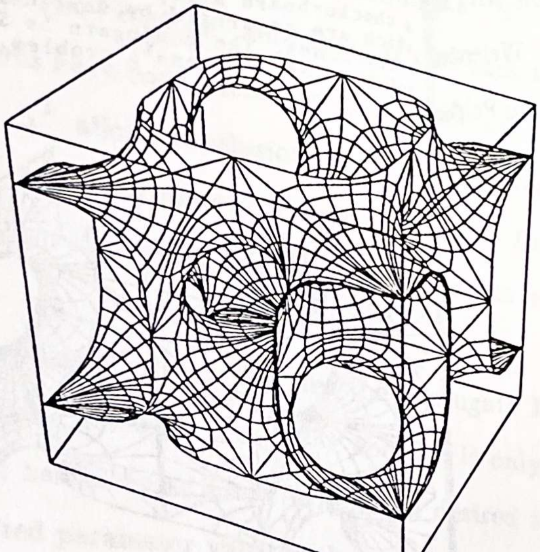
^{2} and M^2_* can be extended by 180°-rotation around the boundary edges sp. reflection in the planes of the boundary arcs. If the angles of the (spheri-al) polygon are of the form $\frac{\pi}{k}$ then the extension can be completed to a com-lete, immersed minimal surface (in S^3).

3.3. Since the eigenvectors of $S_\pm$ and S_* (1.2) are the same, geodesic curva-re lines on $M^2_* \subset M^3(c-1)$ will also be geodesic curvature lines on $M^2_\pm \subset M^3(c\ \ )$, i.e. planar lines of symmetry, on the completed surface. The gles of those symmetry planes at the vertices of the fundamental piece $M^2_\pm$ re (because the Riemannian metric was not changed) still the same as those of e geodesic Plateau polygon.

3.4. To completely control the symmetry group which is generated by the (sym-etry) planes of the boundary arcs of $M^2_\pm$ one finally has to control the turn f the surface normal along each of the boundary arcs. In the case $M^2 \subset \mathbb{R}^3$ it s a great simplification that the total turn of the normal along each boundary rc depends explicitly on the geodesic polygon contour in S^3 and not on fur-her properties of the Plateau solution M^2 ! (A much more detailed discussion f the Plateau solution is needed if one wants to construct a closed surface in ^{3} or H^3 , see [KPS].)



Schoen's S'-S"-surface (1.7, 5.4.2)



Schoen's I-Wp-surface (5.3.3, 5.4.3, 6.1.4)

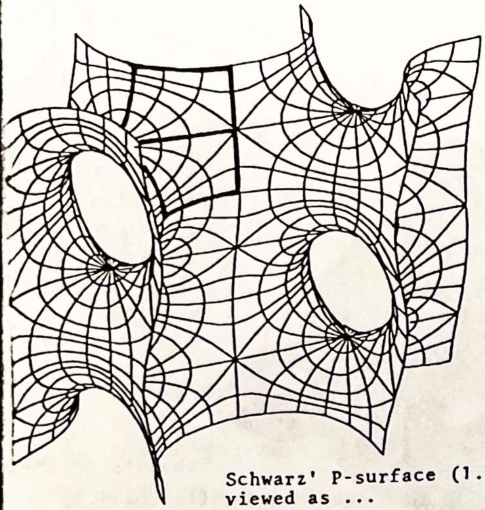
Note that the pattern of symmetry lines on the faces of the cristallographic cell (a quadratic prisma) is the same for the S'-S"- and the I-Wp-surface. The latter has an additional interior ✚-handle.

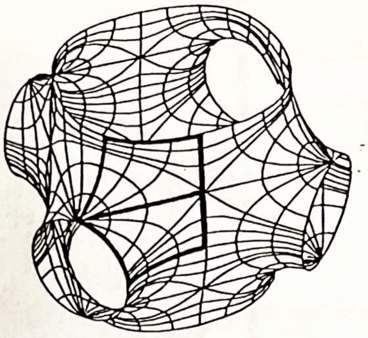
examples which can be obtained
er than I can deal with. The
ems seems to be enormous. - I do
or families of spherical contour
n curvature deformations of the

vn to Schwarz et al. were described
the Weierstraß representation is
ppropriate Riemann surface, we
sentations. In coorperation with
ave worked out global Weier
a few other examples. Konrad Pol
ese formulae. I thank both of them

the spherical polygonal contours

t and new minimal surfaces in S^3

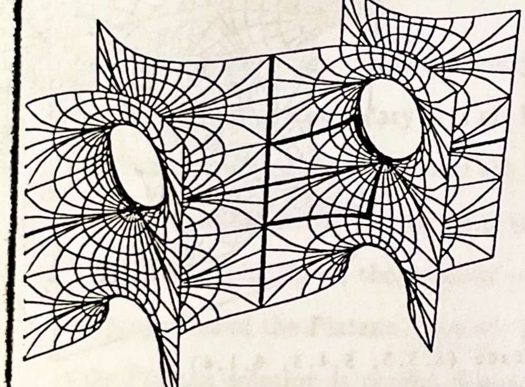


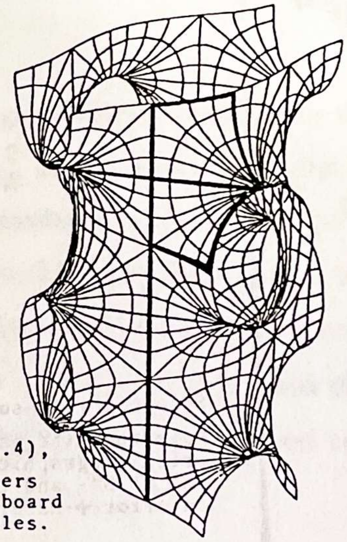


Schwarz' P-surface (1.6.2, 3.6, 5.1.1),
viewed as ...

a checkerboard array of Scherk towers which are separated by vertical symmetry lines. The conjugate is Schwarz' solution of Gergonne's problem.

a checkerboard array of cylinders joined by horizontal handles.





Schwarz' CLP-surface (5.1.4),
viewed as a checkerboard array of Scherk towers separated by vertical lines and as a checkerboard array of cylinders joined by horizontal handles.

metry lines into sufficiently small simply connected pieces, therefore I not deform them into constant mean curvature companions.

conversations with N. Kapouleas the following became clear: Since two great cles usually have four common perpendiculars, two (antipodal) short ones l two long ones, and sometimes infinitely many (on a Clifford torus), one can o find constant mean curvature surfaces with the same symmetry as the isidered minimal surface - which, however, are not obtained by deformation. ith the described conjugate Plateau method I can replace a small neck by a bubble with opposite half necks. Of course one expects also longer pieces of elauney surfaces as "connectors" but the conjugate contours in S^3 get so rge that I do not know how to get "Plateau" solutions. If one uses the fact at Kapouleas' punctured spheres have ends which are asymptotic to Delauney urfaces then the description in S^3 allows conclusions such as: The Kapouleas heres with three ends and 3-fold symmetry have asymptotic Delauney urfaces for which the ratio between the largest and the smallest radius is ≥ 2 .

or two of Schoen's surfaces and many other examples the conjugate Plateau ethod does not work immediately, since the conjugate contour is only determined up to a parameter which has to be adjusted to get the desired surface. e prove existence of the required parameter value with an intermediate value rgument for some height difference. The extreme values of the adjustable parameter correspond to two simpler minimal surfaces which one knows already. hey have opposite signs of that height difference. Continuity in the parameter ollows from work of Nitsche [N]: The conjugate contour can be rotated so that he interior of its Plateau solution is a graph over a convex polygon and the oundary data are piecewise linear with finitely many jumps (corresponding to ertical segments of the contour). Such "Nitsche graphs" depend continuously

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THE TRIPLY PERIODIC MINIMAL SURFACES OF ALAN SCHOEN AND THEIR CONSTANT MEAN CURVATURE COMPANIONS

Hermann Karcher

Dedicated to Wilhelm Klingenberg

We prove existence of Schoen's and other triply periodic minimal surfaces via conjugate (polygonal) Plateau problems. The simpler of these minimal surfaces can be deformed into constant mean curvature surfaces by solving analogous Plateau problems in S^3. The required contours in S^3 are obtained by working with the great circle orbits of Hopf S^1-actions in the same way as with families of parallel lines in $\mathbb{R}^3$. Annular Plateau problems give new embedded minimal surfaces in S^3. For many of the minimal surfaces in $\mathbb{R}^3$ global Weierstraß representations are derived.

Introduction.

In a 1970 NASA report Alan H. Schoen [S] described 17 triply periodic minimal surfaces in $\mathbb{R}^3$ including and inspired by five such examples by H.A. Schwarz and his students. The description was given in terms of "separating surfaces" between two "skelettal graphs"; it did not contain too many mathematical details. Maybe because of the unusual terminology Schoen's work was largely ignored by mathematicians; fortunately, it became widely known in neighbouring sciences, see [A,H,FK,AHLL] and their references.

Hi David:

I have long admired the elegant websites you and Jim Hoffman continue to create on triply periodic minimal surfaces. The graphics are stunning, and the text is clear and beautifully written. Permit me, however, to mention a minor quibble.

Perhaps I'm just excessively thin skinned, but for a long time, I have felt somewhat slighted by the fact that in your website essay on 'skeletal graphs of triply periodic surfaces', there is no hint that the concept of skeletal graphs originated with me. That omission seems unprofessional to me. It's true that it pales into insignificance in comparison with some other egregious appropriations of my ideas (for example, Gandy's plagiarism of whole paragraphs of my NASA Technical Note, in an otherwise elegant paper that simplified the equations for the gyroid). Nevertheless it rankles! Exploiting Coxeter's seven kaleidoscopic cells, the concept of dual skeletal graphs, the idea of grafting on tubules ('handles') to TPMS to increase the genus (to name a few of my original ideas) have -- I believe -- stood the test of time. I arrived at those ideas only after several years of intermittent but intense 'hobby' study of polyhedra, periodic graphs, space groups, and -- beginning in 1966 -- Schwarz's Collected Works. I regret that I failed to explain some of those ideas clearly enough in the Tech Note.

When I gave an embryonic version of the Tech Note to Dirk Struik, in early 1970 or late 1969, he showed it to Dennis Sullivan, who was sharing an MIT office with him. Sullivan invited me to his office, to tell me that he had read my stuff and that he found the discussion of dual skeletal graphs to be particularly interesting, because it is an example of the Alexander-Pontryagin duality. As I believe I made clear in the Tech Note, I never claimed to have developed a rigorous basis for the concept of dual skeletal graphs, but I certainly did recognize its usefulness as an heuristic device for describing both TPMS and their non-zero but constant mean curvature correlates. (I never at any time believed in the possibility of a rigorous generalization to R3 of the theorem that says that every planar graph has a unique dual.)

I am aware that some of the terminology in the Tech Note is archaic (some of it is just plain wrong!), and that some mathematicians -- like Karcher, for example -- profess to have difficulty understanding just what it is I was trying to say, especially in Sections V and VI. I admit that I myself have always recognized that the writing is prolix and difficult to read, and that much of the text is poorly organized. Almost all of it was written -- in great haste -- immediately after we were informed by the director of NASA, on Dec. 30, 1969, that the NASA Electronic Research Center was about to be dissolved, step by step. The Tech Note was typed by the last remaining typist in our division, and there was no opportunity for editing or rewriting, or for showing the manuscript to anyone for criticism.

I don't expect to be cited every time someone mentions the gyroid, but I believe my complaint here is not unreasonable. Do you agree?

<u>Change of subject</u>:

I have spent the past month preparing to respond to a State of Illinois Capital Development Board Art-in-Architecture RFP, concerning a $60,000 grant for an environmental sculpture to be placed just north of SIU's new engineering building. It is open only to 'registered Illinois artists'. I succeeded in getting myself registered. If I can next make it into the initial 'short list', I may have a chance of winning with the final jury, which is composed only of SIU engineering faculty, but there are some serious obstacles. The two state officials who prepare the short list are a former sculptor and former painter. They have never before dealt with a physicist/mathematician, and they have made it plain that I can qualify only if I ally myself with a local sculptor! I have found one, and he is willing to collaborate, but he wants to break up the gyroid into pieces that are not related by any symmetry operation! He finds the pure unadulterated gyroid to be too "60ish" and too "modular"(!). He's a nice guy and a capable sculptor, but he knows nothing at all about either science or mathematics. There are no other sculptors around here who are even remotely qualified. I really do need him, but I'm not sure what's going to happen next. I would rather ditch the whole thing than collaborate on a mish-mash version of the gyroid that is broken into pieces.

I'm planning on having accurate dies made from Mathematica data from the Weierstrass equations, with

29 cast aluminum hexagonal modules (~30" across) welded together in a 12' high columnar tower structure, with a [111] axis vertical. The azimuthal orientation will be chosen to allow the sun's rays to be precisely parallel to the axes of one set of cube-axis tunnels at a certain hour on the autumnal equinox. This will produce a square array of almost perfectly circular bright disks projected orthogonally onto an oblique ground lane. The supporting base will be a flat concrete slab, tiled in either the Poincare hyperbolic map (Coxeter's {6,4!4}), or in the [111] stereographic projection of the tiling of the sphere by Schwarz triangles. I've already got bids for the dies that are well within budget. The sculptor Erwin Hauer, at Yale, has been my adviser. He began to make TPMS-like aluminum and bronze sculptures since the early 60s. (He's been a warm friend since 1971.)

Another change of subject:

I assume you did attend the Blaine Lawson Festschrift. You never acknowledged my long e-mail (after your phone call) about the discovery of the gyroid, and about Blaine's role. What did you think of what I wrote to you?

Regards,
Alan

Mathematica program that will treat every case in nice automatic fashion, just the way I did several years ago when I made the Penrose-like pattern of the 'RHOMBBURST' poster I believe I gave you during our last visit. (If I didn't give you one, let me know, and I'll send you one.)

I would prefer to finish my Mathematica program before I circulate this rhombic filagree stuff. My abstract won't appear until the proceedings of this summer's Bridge conference are published. I learned in 1988 that in the field of computational geometry, less than optimal algorithms recieve only grudging respect, and my not-yet-automated computer rograms are distinctly less than optimal, even though the basis for the algorithm is as elegantly simple as de Bruijn's multigrids and Coxeter's rhombic rosettes.

written -- we should write a joint paper. He agreed, and he even sent me a hurried (but elegant) draft of the outline for such a paper.

Unfortunately, that's where the matter ended. Three months after I returned from Tbilisi, the director of NASA came to Cambridge to announce that Nixon had decided to close the NASA Electronics Research Center, and we were all out on the street six months later. I moved back to California (Cal Arts), where I had a trubulent three years, ending in separation, divorce, and an initially disappointing move to Southern Illinois.

I have made several abortive attempts to do right by my work. Three years ago, a Springer editor encouraged me (not for the first time) to write a book about the subject. I briefly considered doing that, with Ken Brakke as co-author. But even though Ken and I have collaborated on bringing to life a number of old examples whose existence I conjectured between 1968 and 1972 (some -- but not all -- f them are shown on his web pages), I don't have the urge to write anything further about the subject. I realize that I probably have a somewhat curious reputation in the field as what Karcher calls (with just a touch of condescension, perhaps?) an 'auto-didact', but I am resonably satisfied. I was only an amateur, and probably that -- together with a lifelong interest in physics, drawing, carpentry, and model building -- was a major factor in my having been able to see connections that professionals had overlooked. But I will always regret having made a mess of things with Blaine back in 1969.

In conclusion, I have to say that it's a free country, and of course you can say whatever you please, but I would prefer that you not mention this gyroid affair at Blaine's Festschrift. You'll have to judge for yourself whether I am actually as truthful and objective as I claim to be. If you do bring it up, I'm afraid that there may be some skeptics who don't know much about the matter but conclude somehow that I'm just rationalizing (or even lying).

Genug!

Even though you said you would like to see examples of my current work (on what I am calling 'rhombic filagrees') after I have written something more than an abstract, I will shortly mail to you some preliminary stuff, including the abstract I sent off last week to the July Towson 'Bridges' conference. I won't have time to write up a full account of this work for a few months, since I have to return to some nagging patent application matters concerning the cube puzzles I invented three years ago. (I'll send you one member of that family of puzzles, too.) In less than a month, I will join my wife in a trip to Japan, and my work on rhombic filagrees will be in limbo for at least six weeks. I was inspired partly by de Bruijn's theory of Penrose quasicrystals, but also by Ira Gessel's letter to me 20 years ago about a similar dual graph algorithm for constructing what I call 'rhombic rosettes', tilings of regular 2n-gons by rhombs, implied by Coxeter in 'Regular Polytopes' and elsewhere.

Each of the examples of rhombic filagrees I have constrcuted so far (with Mathematica and a new, much faster, computer!) involved laborious work. But I am writing a generic

College, In Norton, MA). From him I learned more about these maps. He steered me to the Coxeter-Moser book 'Generators and Relations for Discrete Groups' (cited in my technical note).

I hope my narration of all these details does not have the effect of just creating suspicion in your mind that I am being defensive because I feel guilty. The only thing I feel guilty about is my having rashly proposed to Blaine, immediately after my Eureka moment, that we co-publish. I remember very clearly telling him that because his question had obviously been the trigger for my sudden insight, there was no way of knowing how long it would have taken me to arrive at the same conclusion if he hadn't asked that question.

Blaine was quite gracious about the whole matter. It's true that he did write to me, shortly after I retracted my proposal of joint authorship, suggesting that in the future I should be more careful about what I proposed and to whom. But I really did not learn anything at all about minimal surfaces from Blaine. I wouldn't be surprised if at that time Bob Osserman believed otherwise, but it's true. When Blaine and I began our telephone correspondence months earlier, he didn't know anytihing at all about the Schwarz surfaces, and he appeared to know next to nothing about the associate surface transformation. I suggested to him that correlates of P and D with constant non-zero mean curvature must surely exist. This was before his beautiful work on the subject.

Early in the course of our telephone correspondence -- it occurred to me one day, after reading a two-page comment by Kummer (Schwarz's father-in-law!) in Schwarz's collected works, that I could define additional triply periodic minimal surfaces by considering the seven Coxeter kaleidoscopic cells and all the skew (straight-edged) polygons whose edges are orthogonal to the faces of those cells. I told Blaine about this idea. He understood it perfectly, and he encouraged me to go ahead with it. He also assured me that my ideas about hybrid surfaces like O,C-TO were mathematically correct. (For at least one face of the corresponding kaleidoscopic ["Coxeter"] cell, more than one edge of the adjoint skew polygon is orthogonal to the face.) So it is of course true that I found it helpful to discuss minimal surfaces with Blaine. I knew nobody in Cambridge with an interest in minimal surfaces. I used to phone Fred Almgren a lot, especially before my talks with Blaine, but he didn't offer any helpful suggestions. On p. 6 of my tech note, I acknowledged discussions with both Blaine and Hans Nitsche. I hope I do not sound arrogant if I say that I believe they both learned considerably more from me than I did from them!

My NASA boss, Lester Van Atta, knew the whole story of my work on minimal surfaces. That amateurish work was one of the principal reasons he had offered me a job at NASA (in the spring of 1966) in the first place. Van Atta insisted, quite vehemently, that I retract my offer to Blaine to share authorship. I knew he was right. I acutely regretted having impulsively invited Blaine to be a co-author.

Eventually, I proposed to Blaine that after the appearance of an initial short announcement of the gyroid in Proc. Nat. Acad. Sci. -- a publication that never even got

As I told you yesterday, Blaine had just a moment earlier told me that it was essential for him to spend full time completing his PhD dissertation, and that -- reluctantly -- he was going to have to give up working on the problem of the gyroid. I then urged him not to give up, because I felt that the solution was somehow just within reach. I had been working hard for a few weeks on boring computational aspects of the graph collapse transormation, applied to several different infinite graphs, both for a computer-anmated film (cf. pp. 86-90 of my NASA tech note) and also for a patent application. I was intensely focussed on that transformation, but the associate surface transformation also kept coming back into mind, since I was still thinking every day about the gyroid.

Once I finally understood the relation between G and P and D, it seemed incredible to me that it had taken me so long to reach that understanding. By that time, I had long been so familiar with the regular and uniform maps on the three surfaces that it 'should have been obvious' to me that there was a deep connection among the three surfaces. On the other hand, just a couple of weeks earlier, Garrett Birkhoff was one of two faculty members at Harvard who attended my lecture there on the gyroid to the Harvard computer science student club (the gyroid was still not yet authenticated!). I had described in detail the polygon maps on the surfaces, and yet Birkhoff also did not see the connection. For that matter, neither did John Milnor, to whom I had shown my gyroid model one day on the street outside MIT, just before going to Madison. (In fairness to Milnor, whom I ha already visited twice at MIT to ask questions about both differential geometry and elementary topology, I must add that I had not discussed with him the regular and uniiform maps.)

The graph collapse transformation applied to the gyroid is quoted -- with illustrations but no equations -- on pp. 85-89 of my NASA technical note. I had been working almost full time on it for the two or three months before Madison, because NASA had required me to apply for a patent on the application of the transformation to the design of expandable space frames. I was also pursuing, at the same time, ramifications of my earlier published abstract 'Infinite Quasi-Regular Warped Polyhedra and Skewness of Regular Polygons', Not. Amer. Math. Soc., vol. 15, 1968, p. 801. I had been interested for the previous two years in both the two Schwarz surfaces, P and D, and in the Coxeter-Petrie infinite regular skew polyhedra. I had pointed out to Coxeter when I met him in Santa Barbara in the spring of 1966, a year before I moved to NASA, the connection between his skew polyhedra and the two Schwarz surfaces. (I had initially 'discovered' the two Schwarz surfaces in April 1966 while playing with vacuum-formed plastic modules of soap films, a day after meeting the architectural designer, Peter Pearce.) This connection is mentioned in the middle of p.49 of my NASA technical note. Coxeter was so astounded by what I said that he just stared at my models (of P and D and his skew polyhedra) for a minute or two before saying anything. He had not previously known of the existence of the Schwarz surfaces.

My point in mentioning all these matters is to explain that for more than a year before my phone converation with Blaine, I had been thinking a lot about the regular and uniform maps that can be inscribed on P and D. Norman Johnson, a Coxeter PhD, had been a warm friend since 1966. He occasionally visited me in Cambridge (from Wheaton

Hi David:

Reiko (my wife) has asked me to apologize to you for the abrupt way she spoke to you on the telephone yesterday. She had, a moment earlier, been interrupted by an aggressive telephone solictor from AT&T, and she mistakenly assumed that you were the same solicitor!

Thnking further about our phone conversation, I would like to make some additional commentsabout my discovery of the gyroid and -- in particular -- about that 'Eureka' moment when I finally realized that the gyroid is simply a surface associate to Schwarz's P and D that happens to be free of self-intersections.

Perhaps second only to my most often cited piece of work in physics (my discovery in 1956 of the connection between the isotope effect for diffusion and the Bardeen-Herring random walk correlation factor), the gyroid is the major single accomplishment in mathematics of my life. I would be extremely unhappy, to put it mildly, if mathematicians came to suspect that Blaine Lawson contributed to my understanding of the underlying mathematics and should therefore be credited as a co-discoverer. It is true that -- as I told you yesterday -- my instant flash of recognition of the true state of affairs occurred in the course of a phone conversation with Blaine, a few days after my return from Madison, where I had (prematurely) delivered a 10-minute talk on the gyroid. I had optimistically assumed, when I earlier mailed in the abstract for that talk, that by August I would surely know how to prove that the plastic artifact I displayed at my talk was indeed a model of an actual minimal surface. But it turned out that my timing was off by about a week!

I have understood since I was an undergraduate at Yale that the problem of credit for contributions to a discovery in mathematics can be so tenuous that it is wiser to err on the side of 'generosity' and credit even minor contributors than to try to claim everything for oneself. But in the present case, Blaine's contribution was only to repeat to me, in the form of a question, the words that I had just uttered to him. He said:

"Are you saying that the 'graph collapse transformation' and the associate surface transformation are related in some fundamental way?"

I had not said that -- and they are not actually related. It is just a coincidence that points on the edges of regular maps on triply periodic minimal surfaces move on elliptical trajectories when those surfaces are subjected to the graph collapse transformation, but those elliptical trajectories are not the same as the elliptical trajectories of points in Bonnet's associative surface transformation. I understood that difference at the time of the phone conversation. I shouldn't even have mentioned the graph collapse transformation to Blaine, since it's not very easy to explain it clearly over the phone. I suppose I must have mentioned it just because I was concentrating on it so intensely at the time of our conversation.