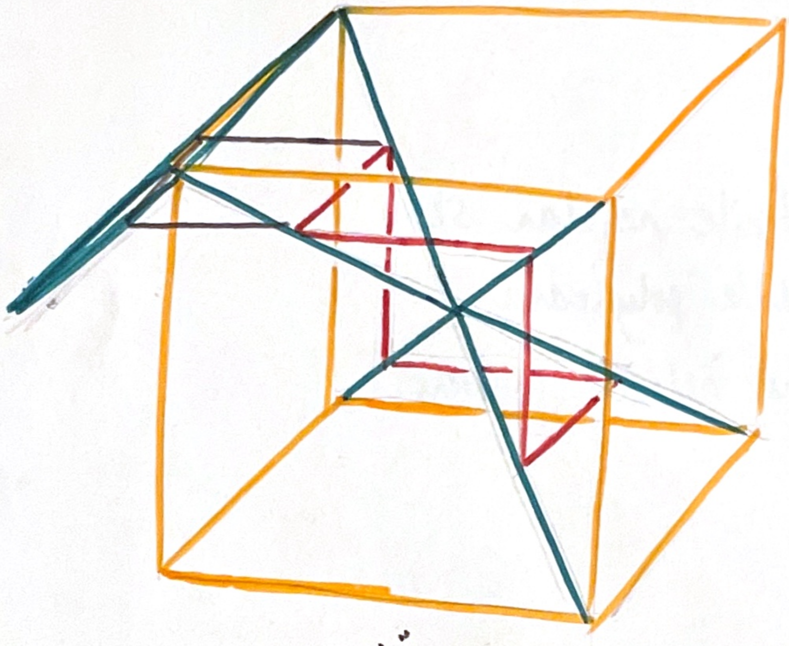
Infinite regular skew saddle polyhedra and related matters.

(face vertex figure) vertex figure "complement"

(polyhedron vertex figure) vertex figure

face



Vertex figure "complement" is square.

Vertex figure is regular [90°] skew hexagon.

Face is [~70.5°] regular skew quadrilateral.

A polyhedron is regular if its faces and vertex figures are all regular.

Regular skew saddle polyhedron

Discovered Feb. 14, 1968

This figure is probably the first example discovered of a saddle polyhedron which is strictly regular (faces are congruent regular [skew] polygons; vertices and edges are all symmetrically equivalent). It is related to the [infinite] regular skew polyhedra of Coxeter and also to the Schwarz infinite periodic minimal surfaces (the two which are based on the 6-connected f.c.c. graph — one generated by spanning the quadrilaterals with minimal surfaces and the other by spanning the hexagons with minimal surfaces). These two Schwarz surfaces can be thought of as infinite regular skew polyhedra of the Coxeter type, but — so long as the elementary skew polygons defined by the edge circuits of the underlying graph [60° quadrilateral for the "diamond" IPMS and 60° hexagon for the "simple cubic" IPMS] are spanned by <u>minimal</u> surfaces, there are no discontinuities in the surfaces across the edges of the skew polygons [asymptotics of the surface], and therefore it would be artificial to regard them as polyhedra, which convention requires to exhibit non-vanishing dihedral angles, between adjoining faces, at the polyhedron <u>edges</u>.

Thus, until now the situation has been that any infinite periodic minimal surface made up of elementary modules which are regular skew polygons is smooth, and therefore not to be thought of as a polyhedron.

If we consider the relationship between symmetric nets and space-filling polyhedra, we cannot fail to be struck by the following fact:

Both the regular skew polyhedra of Coxeter and also the two Schwarz surfaces can be generated by constructing a space-filling assembly of Dirichlet cells [the three Coxeter skew polyhedra] or interstitial domains [the two Schwarz surfaces] corresponding to homogeneous isotropic nets, and then removing all the replicas of one type of face from the polyhedra in the space-filling assembly. The remaining faces constitute the polyhedron or surface, in each case. Thus, the truncated octahedron generates one Coxeter figure, by the removal of the square faces. It is the Dirichlet cell of the 4-valent (110) net. The rhombic tetrahedron (a poor name!), the Dirichlet cell of the diamond net, generates a second Coxeter figure, by the removal of the isosceles triangles. The cube capped by two opposing pyramids,

$(111)_4 - (100)_6$ [2 s.c. + 2 diamond nets]

~~[illegible]~~ homogeneous ~~and~~ anizotropic net $\boxed{Z = 10}$

The interstitial domain is the trigonal 6-faced starfish polyhedron

The symmetry domain is the tetrahedral 10-hedron.

This net is interstitial with respect to the 6-connected f.c.c. net which is the Dirichlet cell of the 4-valent interstitial bcc (or IBCC) net, generates the Expo-Habitat-like agglomeration of defective cubes in close-packing by the removal of the ~~triangle~~ faces. The two Schwarz surfaces can be generated from a space-filling of saddle 10-gons by removing either the quadrilaterals (giving the ~~diamond~~ simple cubic IPMS) or the hexagons (giving the diamond IPMS). This 10-gon may also be regarded as the symmetry domain of the compound net $4_{111}-6_{100}$ which consists of 2 interpenetrating S.C. nets joined by 2 interpenetrating diamond nets. (The Dirichlet cell of such a net is the truncated octahedron.)

The regular skew saddle polyhedron discussed here may be thought of as arising from a space-filling of the interstitial domains of the b.c.c. net, with three of the four reg. skew quad. faces removed from each domain. Unfortunately, this is not a very useful way to describe it, because it affords no clue as to which three out of four faces are supposed to be removed. A more useful way to describe it is based on the 6-connected b.c.c. net.

Correction: It is useful to describe this figure in terms of the interstitial domains of the b.c.c. net. The rule is: Remove only those quadrilateral spanning surfaces, from the assembly of b.c.c. int. domains, which include one of the edges removed to transform the 8-connected b.c.c. net

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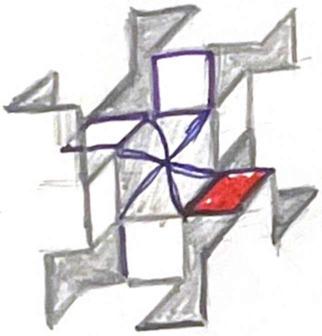
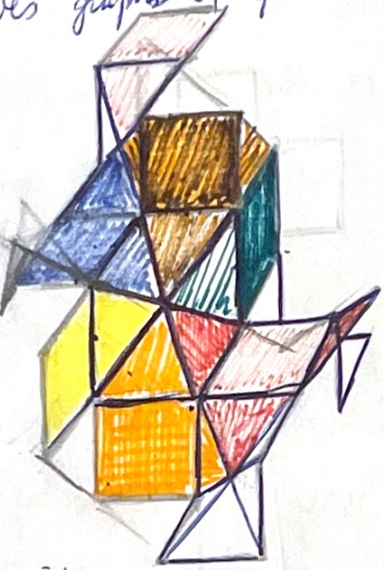
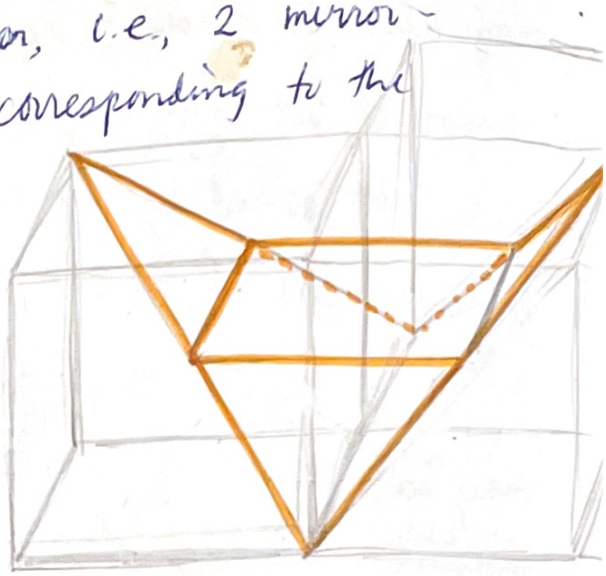
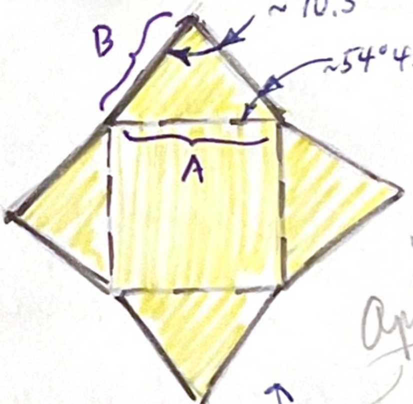
Note: If a vertex figure (90° regular skew hexagon) is inscribed around each vertex and spanned by a minimal surface, and the vertex figure complements (squares) are also spanned, then we can construct a $\boxed{\text{uniform skew saddle polyhedron}}$ [all faces regular, all edges and vertices symmetrically equivalent]. It will have the same overall topological structure as this regular skew saddle polyhedron, i.e., 2 mirror-symmetric interpenetrating labyrinths corresponding to the LH & RH Laves graphs of girth 10.









$$\frac{A}{B} = \frac{2}{\sqrt{3}}$$

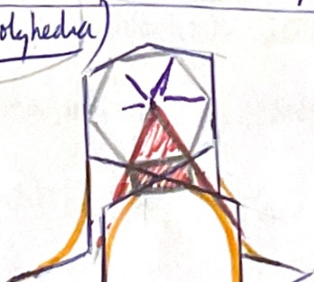
This suggests a simple way to construct an approximate model of this figure with ordinary paper. Each regular skew saddle quadrilateral can be approximated by folding (45°) the isosceles triangle tabs of this figure.

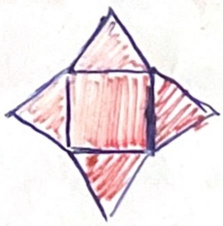
N. Johnson
Apeirons
Canad. J. Math
early 50's?

(Infinite Uniform Skew Polyhedra)

Also squares & hexagons

Similarly, a square tabbed with equilateral triangles serves as a substitute for the 60° regular skew quadrilateral in the diamond IPMS construction. (I have used such "rectified" paper models before.)

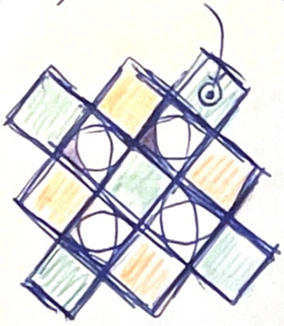




into the homogeneous isotropic 6-connected b.c.c. net.
(Consider the contrasting behavior of the 6-connected f.c.c. net and the 3-connected s.c. net: the removal of 6 and 3 edges from the original 12 and 6, respectively, eliminates all of the original spanned edge-circuits [triangles and squares, respectively] and leads to new spanned edge-circuits in the generation of the interstitial domains: 60°-quads and 60° hex., in 6-connected f.c.c., and the monstrous-looking 10-gon in the 3-connected s.c. net. Thus, the 6-connected b.c.c. net has the topological peculiarity of preserving 1/4 of the original spanning-surfaces of the interstitial domains of the 8-connected b.c.c., destroying the closed-cells character of the assembly of interstitial domains, and yet not providing any other edge-circuits which can be spanned by minimal surfaces without creating unwanted intersections (i.e., intersections which do not lie along edges of the 6-connected net). [For that matter, any choice of edge-circuit polygon for spanning leads to intersections not restricted to lie along edges of the 8-connected net!]

The modules in this figure are related to one another by 120° rotation of each into its replicas, with respect to each edge. (The sense of this rotation is constrained by the relative orientations required for (111) edges.)

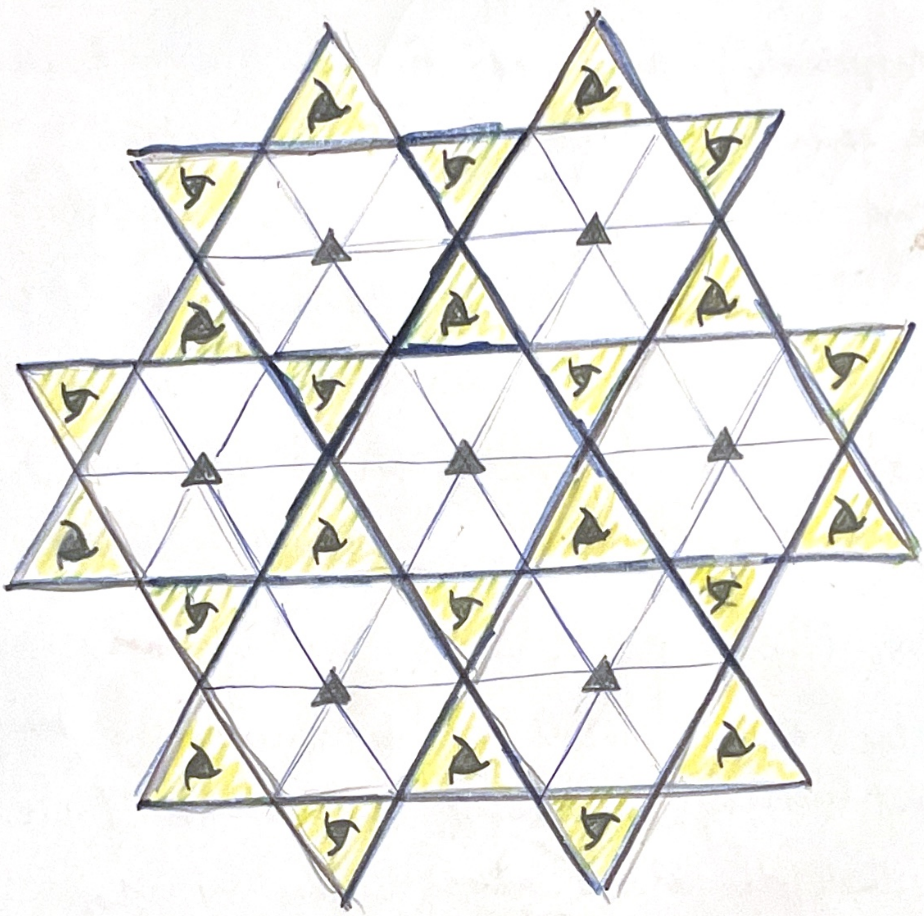
6) R — the [100] axis through the center of each face — is a $(\bar{4})$ axis.



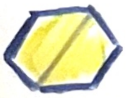
7) S — the [111] axis symmetrically passing through each vertex — is a $(\bar{6})$ axis.

R and S are the symmetry elements common to all regular skew saddle polyhedra.

tunnel



5) The 6 sets of (110) axes ~~are~~ which lie centered on the finite "rhombic" tunnels are axes of 2-fold symmetry. (These axes are $\perp$ to the edges ~~center~~ at edge center positions.)



tunnel

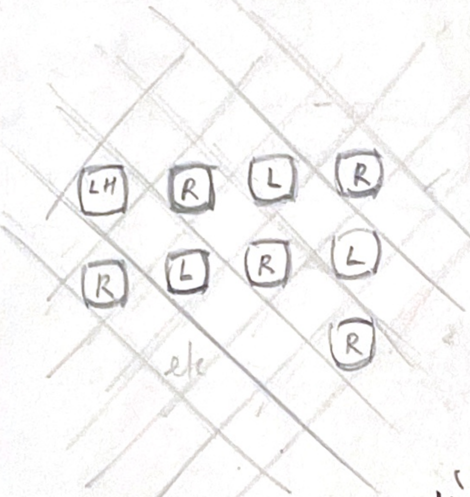
The space group defined by this distribution of symmetry elements is the same as that describing the superposition of the 2 (LH & RH) laves graphs of girth 10.

To identify the space group of this figure, let us list the symmetry elements. The space lattice is b.c.c.

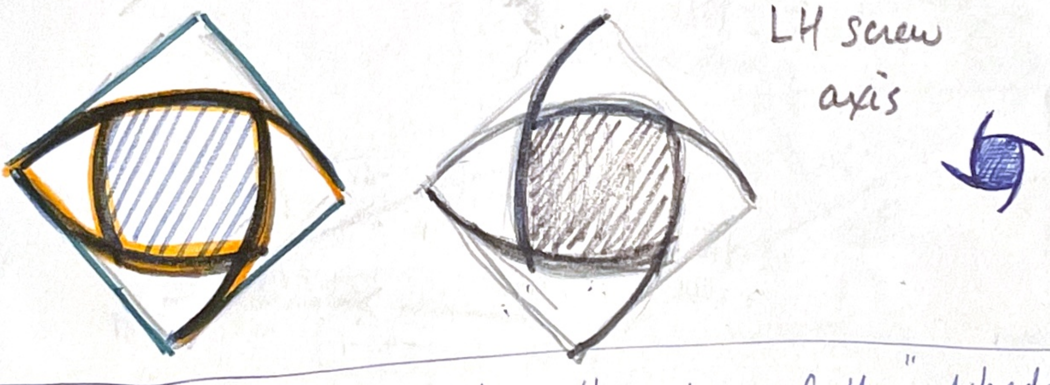
1) Each vertex is a center of inversion.

2) The (111) axes along the missing edges are 3-fold axes

3) The 3 orthogonal sets of (100) axes centered on the open tunnels (which have 4-fold rotational symmetry in projected view) are 90° screw axes (LH in one labyrinth and RH in the other).



Note that these tunnels, seen in projection, automatically provide parts of the crystallographic symbols for four-fold screw axes of the correct handedness in each set of tunnels.



4) The (111) axes which lie along the edges of the "polyhedron" are not symmetry axes, (except as axes of "rotatory reflection", i.e., inversion) apparently, but all the (111) axes (4 sets) which are centered on the open triangular tunnels (which have 3-fold rotational symmetry in projected view) are 120° screw axes (LH in one labyrinth and RH in the other).

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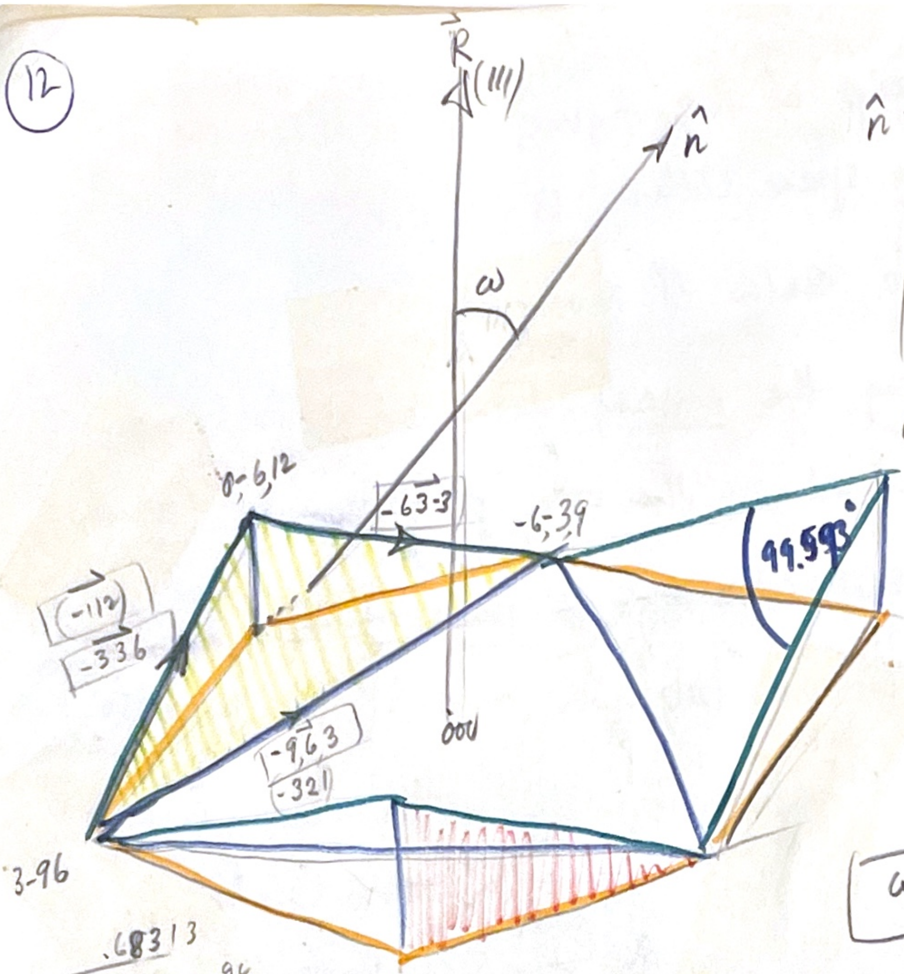


$$\hat{n} = (\bar{3}21) \times (\bar{1}12)$$

$$\begin{vmatrix} i & j & k \\ -3 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix} = (3\,5\,{-1})$$

$$\omega = \cos^{-1}\left(\frac{R \cdot n}{|R||n|}\right)$$

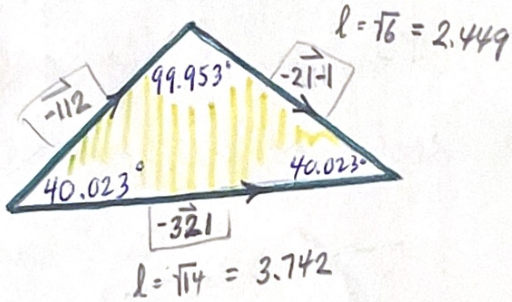
$$= \cos^{-1}\left[\frac{(111)\cdot(35\text{-}1)}{\sqrt{3}\,\sqrt{35}}\right]$$

$$= \cos^{-1}\left\{\frac{3+5-1}{\sqrt{3}\,\sqrt{7\cdot5}} = \frac{7}{\sqrt{105}} = \frac{7\sqrt{105}}{105} = \frac{\sqrt{105}}{15} = \frac{10.247}{15} = .68313\right\}$$

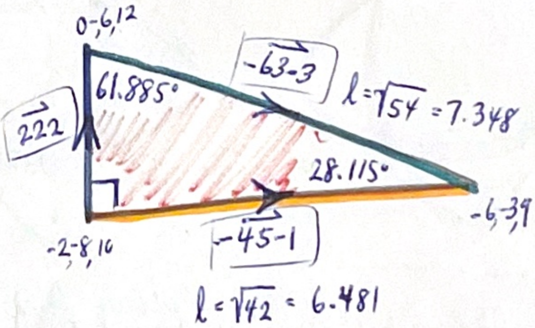
$$\boxed{\omega = 46.912^\circ}$$

$$15 \overline{)\,10.247\,} = .68313$$

90 − 43.088 = 46.912



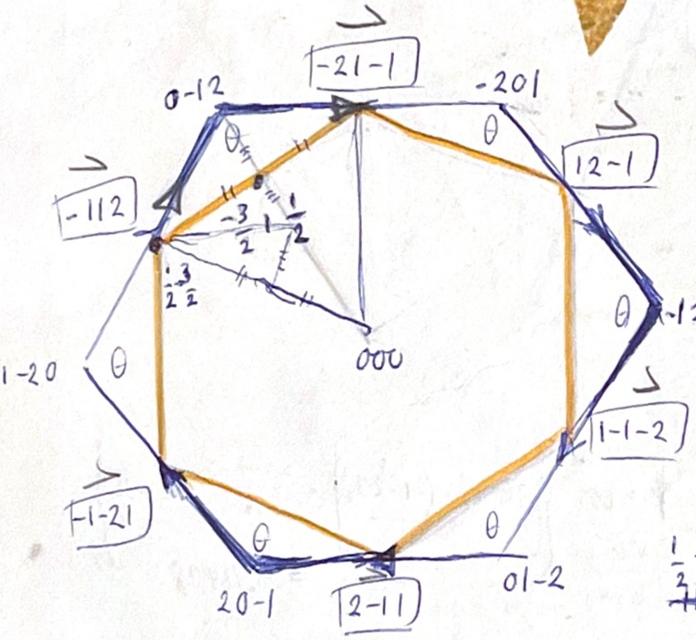
Dimensions of parts of shillelagh for 99.593° regular skewhexagon



(~~reciprocal~~: i.e., mid-edge points of [perpendicular] edges of the two regular polyhedra are common) [Coxeter, Reg Polyt. p. 17)

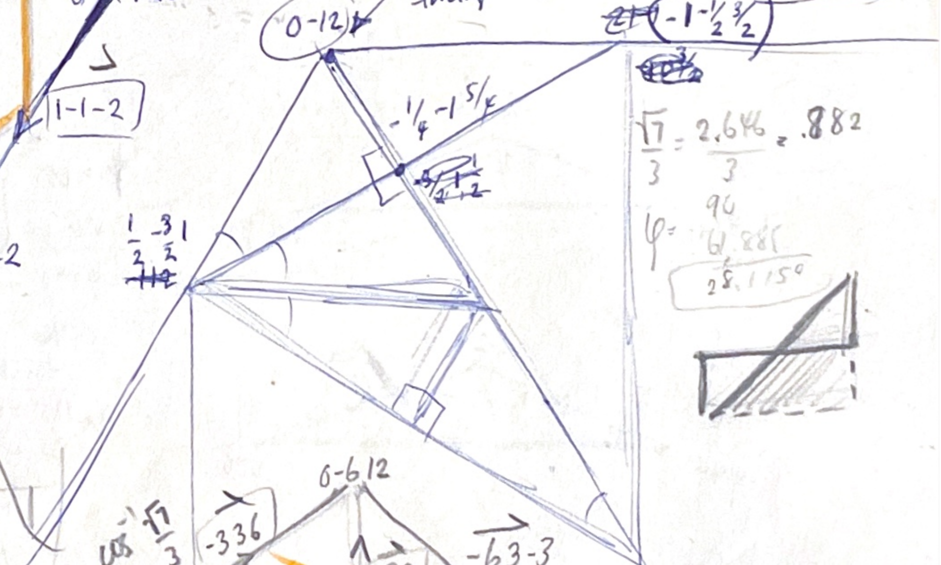
An adjoint figure, based on this one, is found by constructing a regular skew hexagon by joining the midpoints of the regular skew quadrilaterals (cf. the analogous operations for the 2 cubic Schwarz IPMS's).

This shows the arrangement of 6 70.5° regular skew quadrilaterals at each "bumpy monkey saddle" centered on a standard vertex of the infinite regular skew saddle polyhedron.



$\theta = \cos^{-1}\left(-\frac{1}{6}\right) = 99.953°$ 99.5930

projection of (0-12) onto plane (111) through origin = $\left(-\frac{1}{3}, -\frac{4}{3}, \frac{5}{3}\right)$



$\frac{\sqrt{7}}{3} = \frac{2.646}{3} = .882$

$\varphi = 61.885$

$28.115°$

$\frac{4}{3}\left(-\frac{1}{4}, -1, \frac{5}{4}\right)$

$= \left(-\frac{1}{3}, -\frac{4}{3}, \frac{5}{3}\right)$

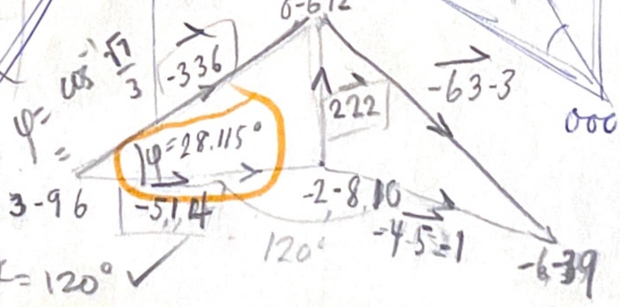
0 -1 2

$-\left(-\frac{1}{3}, -\frac{4}{3}, \frac{5}{3}\right)$

$= \boxed{\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}}$ ✓

$\varphi = \cos^{-1}\frac{\sqrt{7}}{3}$

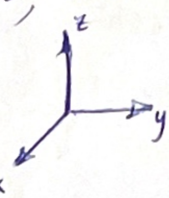
$\varphi = 28.115°$



$(-514)\cdot(4-51) = \frac{-20-5+4}{42} = -\frac{1}{2}$ $\therefore \measuredangle = 120°$ ✓

(14)

Now compute the angle ψ through which each ~~99.453°~~ 99.593° regular skew hexagon is rotated, about each {112} edge, into its replicas

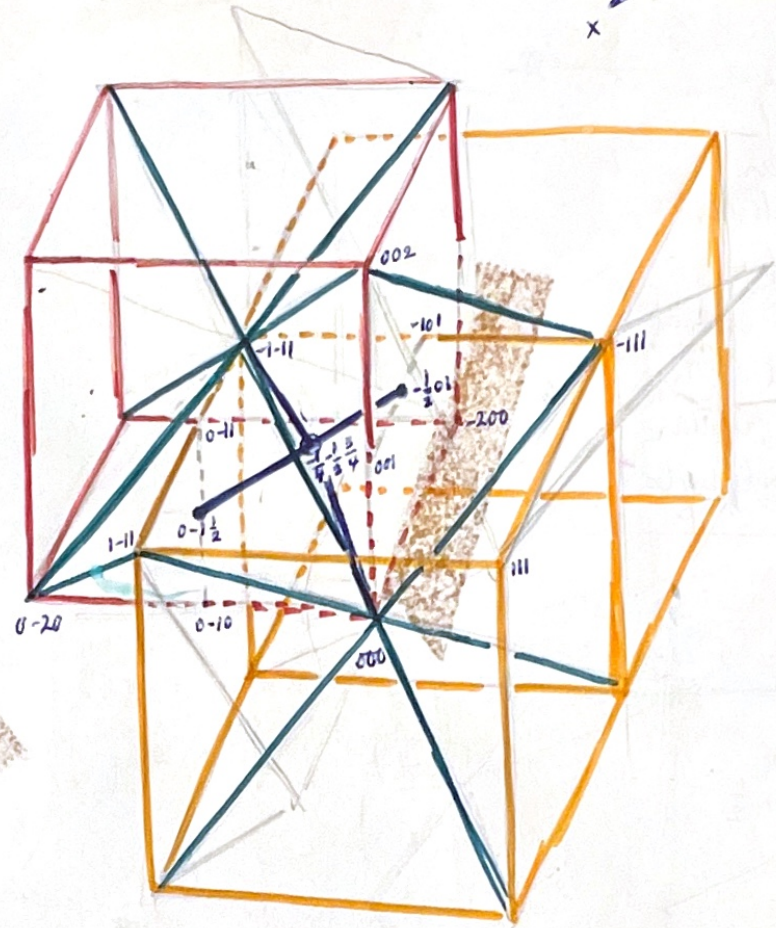


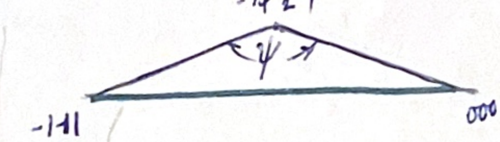
70.5° {4} 2.041
1.414
8164
2041

2.886

2.041

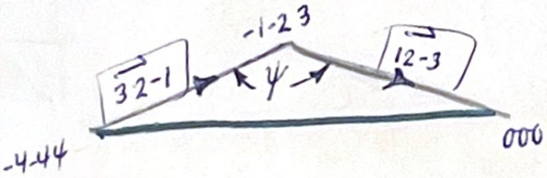
1.443



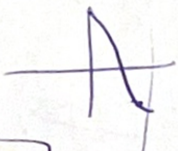


$$\psi = \cos^{-1}\left\{\left[\frac{(32\text{-}1)\cdot(\text{-}1\text{-}23)}{14}\right] = \frac{-3-4-3}{14} = -\frac{5}{7}\right\}$$

$= -.71428$

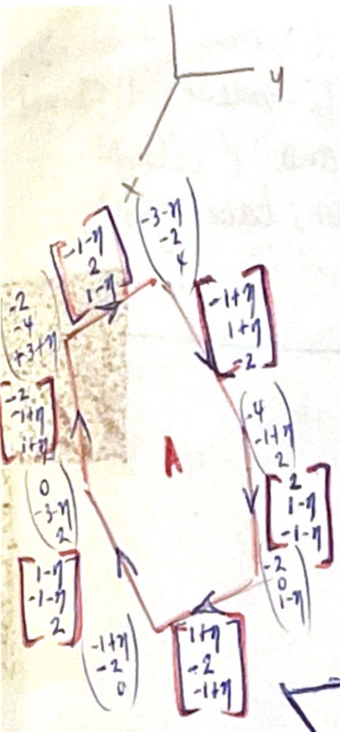


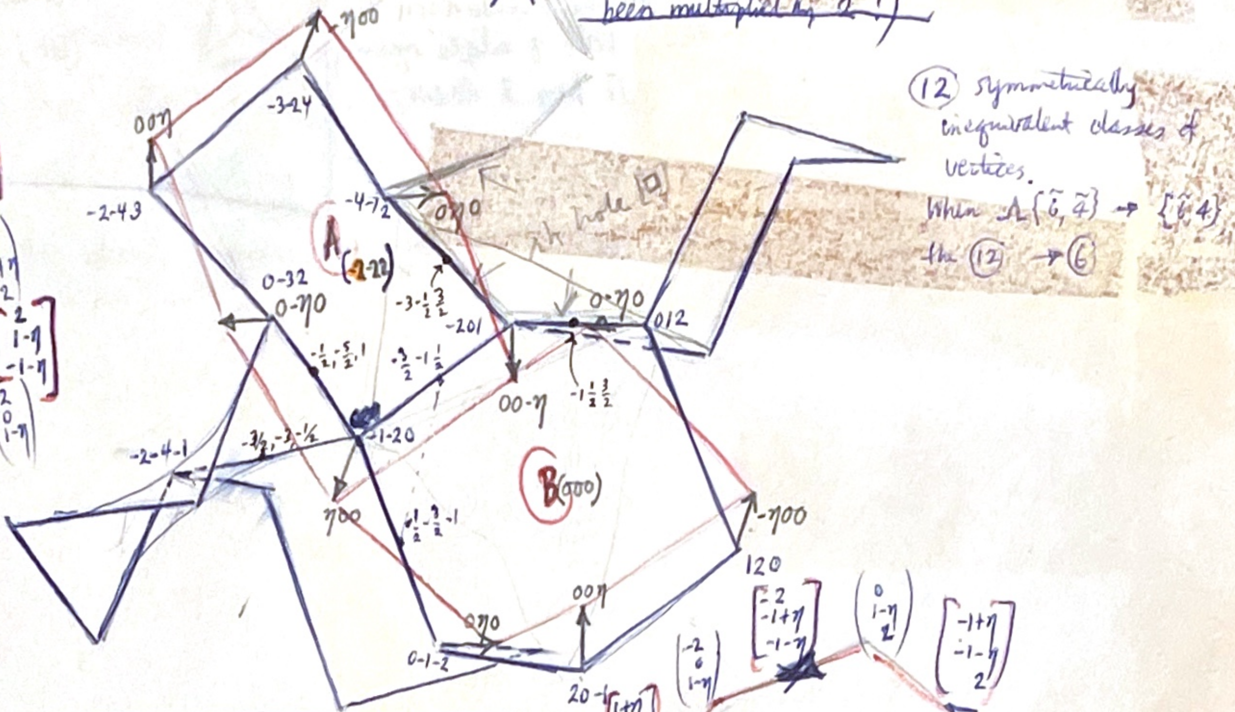
90°
45.584°
135.584°



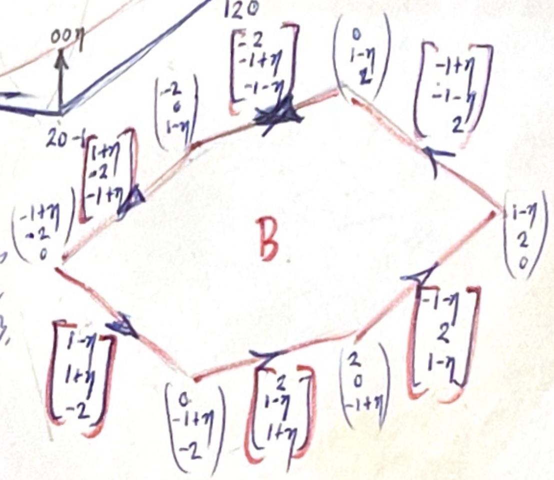
$$\therefore \boxed{\psi = 135.584° \quad \left[= \cos^{-1}\left(-\tfrac{5}{7}\right)\right]}$$

★ (~~For convenience the coordinates of in ink have been multiplied by 2!~~)

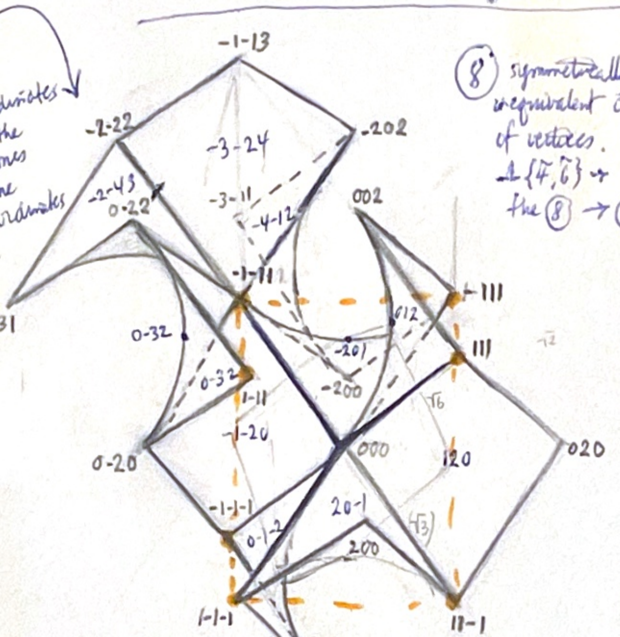




(12) symmetrically inequivalent classes of vertices.
When $A\{\tilde{6},\tilde{4}\} \to \{\tilde{6},4\}$ the (12) → (6)



★ (Here, the inked coordinates are related to the pencilled ones only if the inked coordinates are divided by 2!) ★



(8) symmetrically inequivalent classes of vertices. When $A\{\tilde{4},\tilde{6}\} \to \{\tilde{4},3\}$, the (8) → (4).

$$\cos\varphi_6^{(A)} = \frac{\begin{pmatrix}1+\eta\\-2\\-1+\eta\end{pmatrix}\begin{pmatrix}-1+\eta\\1+\eta\\-2\end{pmatrix}}{(1+\eta)^2+(1-\eta)^2+4} = \frac{-1+\eta^2-2-2\eta+2-2\eta}{2+2\eta^2+4} = \boxed{\frac{\eta^2-4\eta-1}{2\eta^2+6}}$$

$$\cos\varphi_6^{(B)} = \frac{\begin{pmatrix}1-\eta\\1+\eta\\-2\end{pmatrix}\begin{pmatrix}-2\\-1+\eta\\-1-\eta\end{pmatrix}}{2\eta^2+6} = \frac{-2+2\eta+\eta^2-1+2+2\eta}{2\eta^2+6} = \boxed{\frac{\eta^2+4\eta-1}{2\eta^2+6}}$$

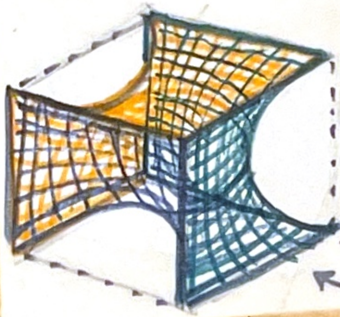
$$\sigma_6^A = \sqrt{\frac{\cos\varphi-\cos\varphi_6}{1-\cos\varphi}} = \left[\frac{\frac{\eta^2-4\eta-1}{2\eta^2+6}+\frac{1}{2}}{1-\frac{\eta^2-4\eta-1}{2\eta^2+6}}\right]^{1/2} = \left[\frac{2\eta^2-8\eta-2+2\eta^2+6}{2\left[2\eta^2+6-\eta^2+4\eta+1\right]}\right]^{1/2} = \left[\frac{4\eta^2-8\eta+4}{2\left[\eta^2+4\eta+7\right]}\right]^{1/2} = \frac{\sqrt{2}\left[\eta^2-2\eta+1\right]^{1/2}}{(\eta^2+4\eta+7)^{1/2}} = \boxed{\frac{\sqrt{2}\,(1-\eta)}{(7+4\eta+\eta^2)^{1/2}}} = \sigma_6^A$$

(no real roots in denominator) $\eta=0) \mid \sqrt{\frac{2}{7}}$

$$\sigma_6^B = \left[\frac{\frac{\eta^2+4\eta-1}{2\eta^2+6}+\frac{1}{2}}{1-\frac{\eta^2+4\eta-1}{2\eta^2+6}}\right]^{1/2} = \left[\frac{2\eta^2+8\eta-2+2\eta^2+6}{2\left[2\eta^2+6-\eta^2-4\eta+1\right]}\right]^{1/2} = \frac{(4\eta^2+8\eta+4)^{1/2}}{\left[2(7-4\eta+\eta^2)\right]^{1/2}} = \boxed{\frac{\sqrt{2}\,(1+\eta)}{(7-4\eta+\eta^2)^{1/2}}} = \sigma_6^B$$

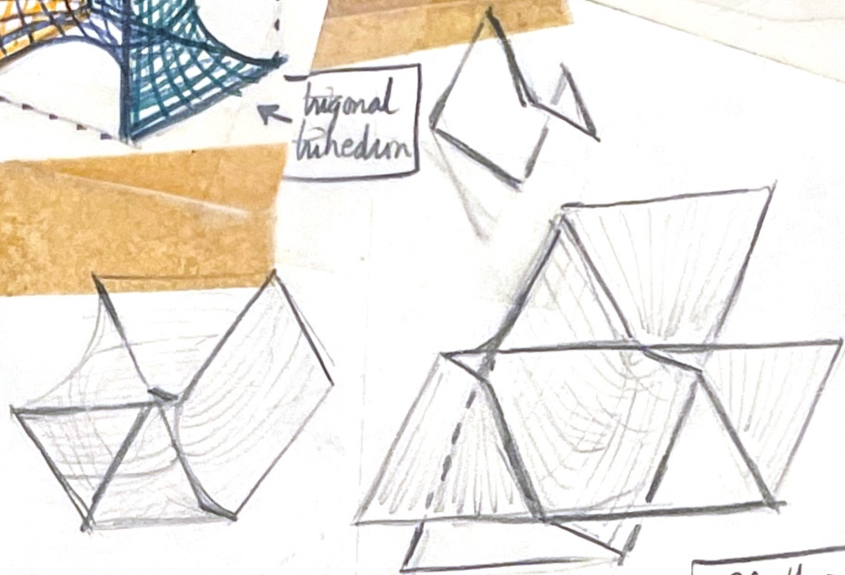
$$\left.\begin{array}{l}\text{When } \eta=1,\ \sigma_6^A = 0\\ \sigma_6^B = \sqrt{2}\ \ (\varphi = 60^\circ)\end{array}\right\} \text{Thus is } \left\{\begin{array}{c}6\\ \tilde{6}\,[\sqrt{2}]_{(60^\circ)}\end{array}\right\}_x = M\{\tilde{6},\tilde{6}\}$$

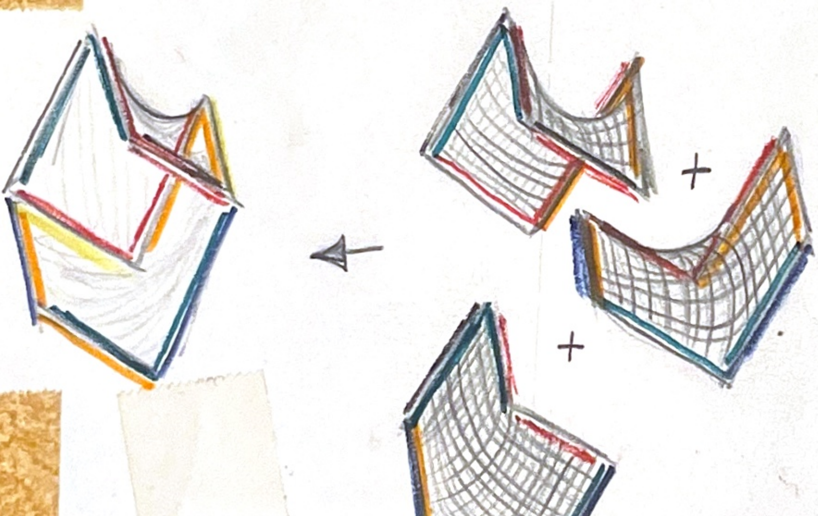
This interstitial domain (and its enantiomorph) is a rhombic hexahedron with 3 edges missing. It has 3 faces



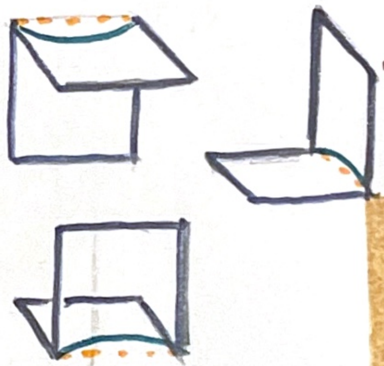
trigonal trihedron

Relation between sites of 6-valent (112)-net and 3-valent (110) Laves graph.





Cf. this familiar "cube" construction

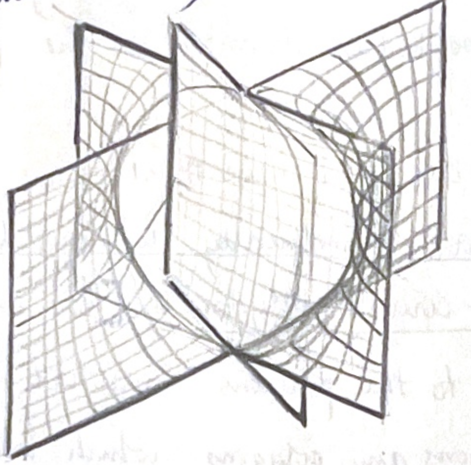


I think I may have discovered a way to construct an assembly of "reasonable" interstitial domains for the 6-connected b.c.c. net — that is, domains which intersect each other only along edges of the net. The problem, until now, has been that the only non-self-intersecting edge-circuit polygons in the net are the $\sim 70.5^\circ$ regular skew quadrilaterals, but their spanning surfaces do not define an infinite assembly of closed cells — instead, they lead to the infinite regular skew saddle polyhedron described in the preceding pages.

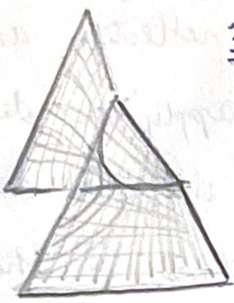
It appears that one can use a device I have tried to use before, but always without success: <u>span only a symmetrically distributed fraction of the replicas of a particular edge-circuit polygon</u>. [We are speaking here of a polygon with full regard to the positions in which it is found. Thus, we count as <u>2 distinct polygons</u> any polygons which happen to be congruent but do not occur in symmetrically equivalent positions (they may, of course, be related by a mirror reflection, and be of opposite handedness).] When I ~~tried~~ to apply this device before, it was in order to increase the <u>size</u> of the interstitial domains, so as to reduce the number of edges and vertices of the resulting symmetry domain, but there was no difficulty about self-intersections to worry about. Here, the problem is more severe. In any case, I believe I can construct a set of interstitial domains without unwanted intersections by omitting a certain fraction ($\frac{1}{2}\,\frac{2}{3}$) of the  skew hexagons I originally made shillelaghs for earlier this week. <u>If</u> this works, it does so by ~~being based~~ virtue of the fact that each Laves girth-10 tunnel (i.e., the <u>L</u>H and the <u>R</u>H) will accommodate the trigonally symmetric

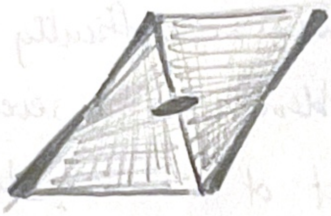
Two of the three kinds of interstitial domains

trigonal
6-faced
starfish
saddle
polyhedron
(1 of the interstitial domains)



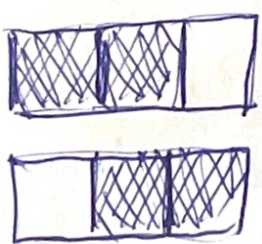
digonal
3-hedron
(one "hinged rhombi" face [a face of the starfish shown above] and 2 ~70.5° reg. skew quadrilaterals)





(top view)

These digonal trihedra have a "degeneracy" of 2 with respect to the set of pairs of quadrilateral faces in the ∞ reg. skew saddle polyhedron against which they fit.



This follows from the screw symmetry of the tunnels, or — put another way — from the availability of 2 subsets of alternate Laves sites on each graph of given handedness.

* (adjacent quadrilateral faces in ∞ polyhedron [schematic])

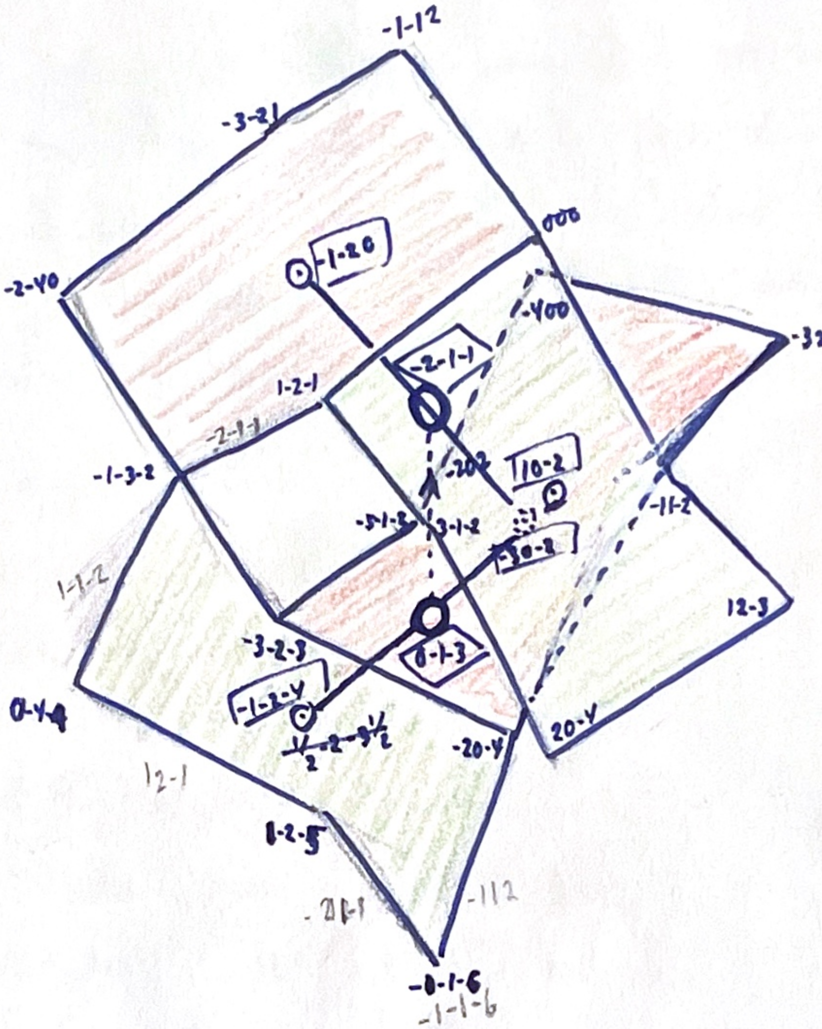
6-faced starfish saddle polyhedra I made earlier this week so long as they are omitted from alternate sites. This amounts to their being distributed on the nodes of the 6-valent (112) net, which can be constructed by connecting alternate ~~nodes~~ nodes of the Laves graph, as indicated in the sketch 3 pp. back.

The arrangement is as follows:

In the LH Laves labyrinth, the starfish are located on alternate nodes of the underlying girth-10 net. On alternating faces of each starfish lie trihedra of the kind shown on the opposite page. The faces of the starfish which are not shared with these [digonal] trihedra are shared with the trigonal trihedra shown 3 pages back ("defective rhombic hexahedron").

In the RH Laves labyrinth, the digonal and trigonal starfish occur in forms enantiomorphic to the forms in the LH labyrinth. The starfish appear on sites which bring faces opposite to those in the other labyrinth to bear on the digonal and trigonal trihedra, respectively.

If I am not mistaken, these polyhedra will serve as interstitial domains. I haven't yet got around to examining the question of uniqueness of this set. I suspect it's not a unique set. Anyway, the next step is to find out what kind of symmetry domain results from all this.


-1-12
-3-21
000
-2-40
-1-20
-400
-32-1
-2-1-1
1-2-1
-2-1-1
-202
10-2
-1-3-2
-3-1-2
3-1-2
-30-2
-11-2
1-1-2
12-3
0-1-3
-3-2-3
-1-2-4
0-4-4
-20-4
20-4
12-1
1-2-5
-112
-21-1
-0-1-6
-1-1-6

I've just realized that I've made a mistake! It's good to know, because it makes everything simpler.

I've been regarding the "hinged rhombi" skew hexagons as being all symmetrically equivalent, overlooking the fact that some of them are close to skew quadrilateral pairs — i.e., they share six edges with such adjacent pairs — while others face large open voids. Now it turns out that the hexagons which share edges with pairs of adjacent quadrilaterals are <u>self-intersecting</u> and therefore should not be spanned. <u>FINE!!</u>

It also appears that the remaining hexagons are <u>not</u> self-intersecting, and therefore the spanning algorithm seems to be working perfectly after all!

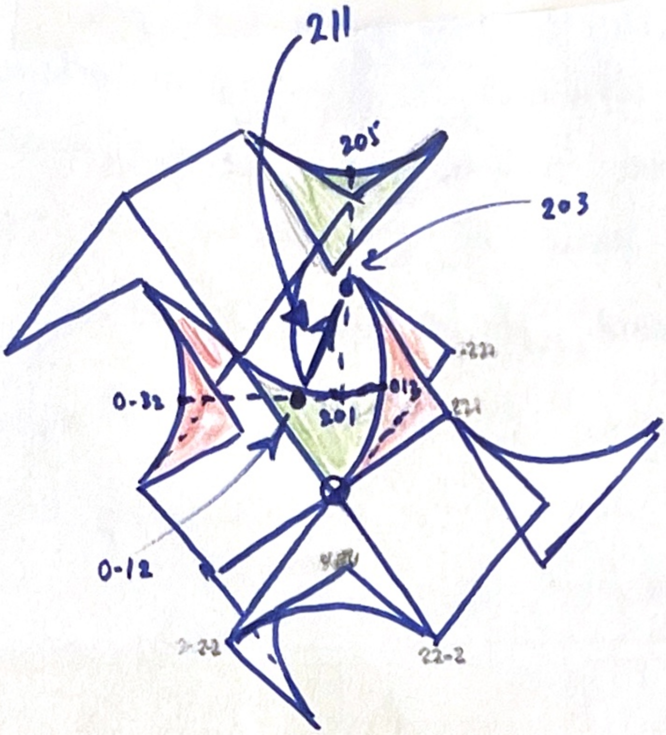
(I've been under the impression since three days ago that the spanning algorithm was KAPUT, because of this refractory net.)

Without working models, I can't be positive, but it's now my impression that each Laves labyrinth is filled with just two kinds of interstitial domains — a modification of the <u>trigonal starfish hexahedron</u> (alternate hexagonal faces having "acquired" two regular quadrilaterals, according to the arrangement exhibited by the digonal trihedron), and the <u>trigonal trihedron</u> which is a kind of defective rhombic hexahedron. I believe this all works strictly according to the algorithm, and now

The "normal" graphs of L are: $\{\tilde{6}, \tilde{4}\}_L$: Laves graph

$\{\tilde{4}, \tilde{6}\}_L$: t{Laves graph}

$\{\tilde{6}, \tilde{6}\}_L$: Laves graph



$$\begin{array}{r} 203 \\ -\ (0\text{-}12) \\ \hline = \boxed{211} \end{array}$$

"Normal" graph of $\{\tilde{4}, \tilde{6}\}_L$ is t{Laves graph}

Any two parallel $\{\tilde{4}\}$'s of the same color have four pairs of parallel faces with the following property: one face of each pair shares an edge with one of the original $\{\tilde{?}\}$'s; the other face shares a vertex.

I suspect it may be unique after all

The starfish polyhedron acquires three additional faces, altogether, and is therefore a 9-hedron.

$\boxed{\text{The 9-hedron and the 3-hedron occupy alternating nodes of each Laves graph.}}$

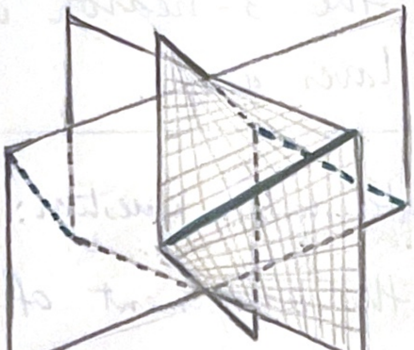
This raises a very interesting question:

How do we decide the assignment of 9-hedron vs. 3-hedron in the second Laves tunnel, after we have filled the first?

Since the nodes of each Laves tunnel (graph) are all exactly equivalent, the filling in one tunnel is arbitrary with respect to 3-hedron sites vs. 9-hedron sites. However, it would seem possible that the way the second tunnel is filled might matter (e.g., the symmetry domains might not have the right point group symmetry after all, with <u>one</u> of the two possible arrangents).

What is possibly even more interesting is just the double degeneracy mentioned earlier. Thus, choosing an <u>initial pair</u> of adjacent quadrilaterals <u>lifts the degeneracy</u>

, and from there on,

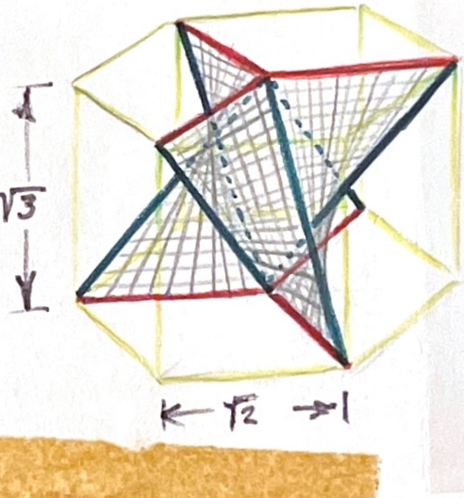


The correct version of the trigonal 9-hedron (+ enantiomorph)

It has trigonal symmetry, without mirror planes.

Connections between centroids of adjacent symmetry domains

Symmetry domain?
(Presumably not!)



all the other quadrilaterals in the ∞ net are assigned in (25)
pairs in a definite way!

Having the flu doesn't seem to be improving my reasoning ability (I've been home the past two days).

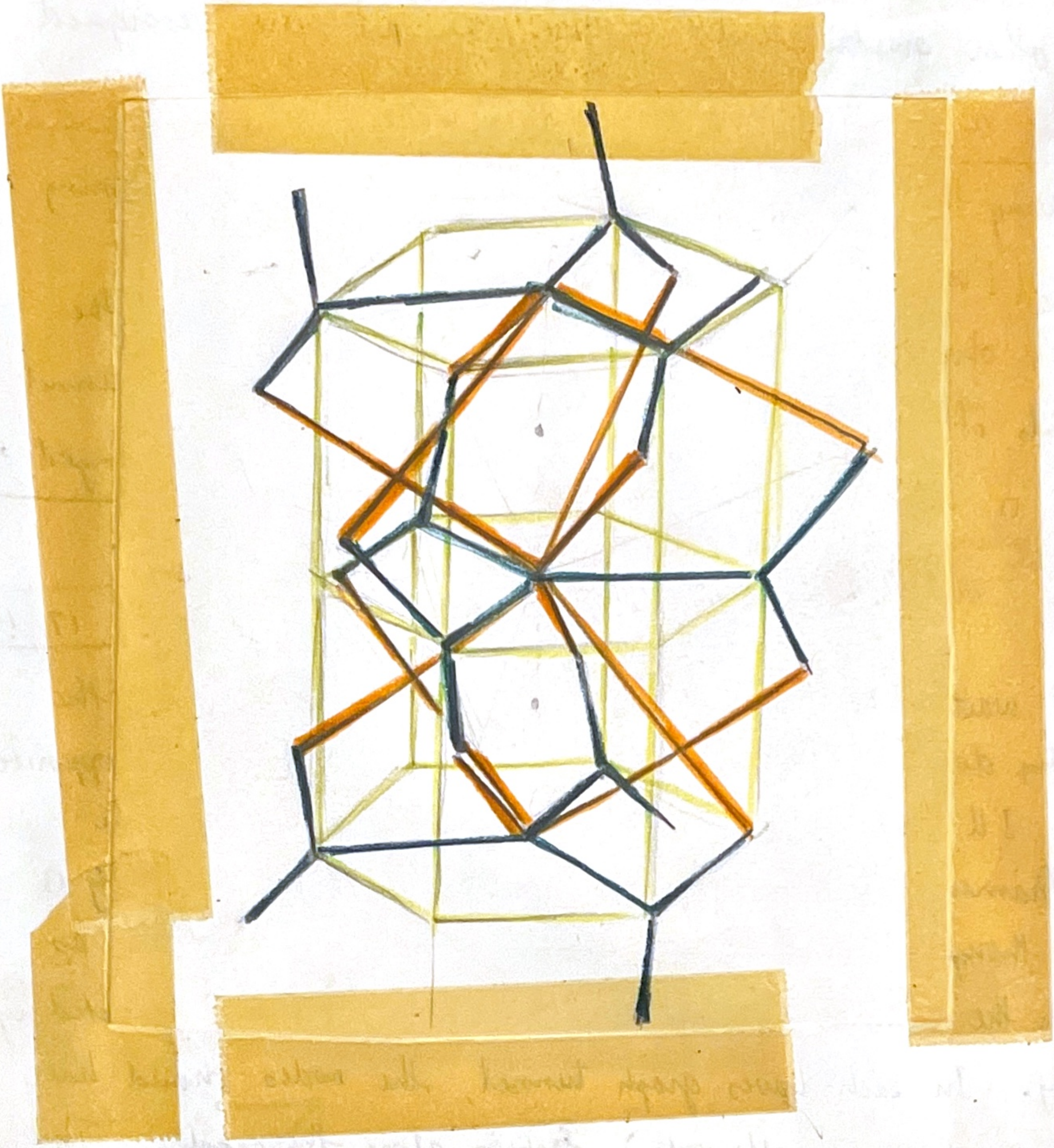
It's obvious that the criterion for the placement of the two kinds of interstitial figures in the second Laves tunnel system is that the symmetry of the net not be changed.

Saturday morning Feb. 17, 1968

Without waiting to verify the details, I have surmised that the symmetry domain is the figure at the lower right on the opposite page. I'll have to construct models (string and yarn in a s.c. framework) to verify it. Somehow, I suspect something is wrong, though. Now that I think about it, it doesn't look as though the centroids of the interstitial domains are connected up properly. In each Laves graph tunnel, the nodes should have 3 and 9 edges, in alternating fashion along the graph. This isn't the way I've done it.

The guess shown on the next page looks more plausible. It appears to satisfy the alternating 3- and 9- connectedness of centroids of the interstitial domains. Each 9-hedron centroid is connected via a (110) edge to 3 neighbors on the same tunnel and ~~to~~ to 3 neighbors each, from the other tunnel, "above" and "below" the coordination plane of origin of the 9-hedron. Each 3-hedron centroid, on the other hand, is connected only to its 3 immediate neighbors, via (110) edges. I have assumed here that the 9-hedra ~~at~~ from ~~at~~ adjacent enantiomorphic tunnels are simply stacked on top

6



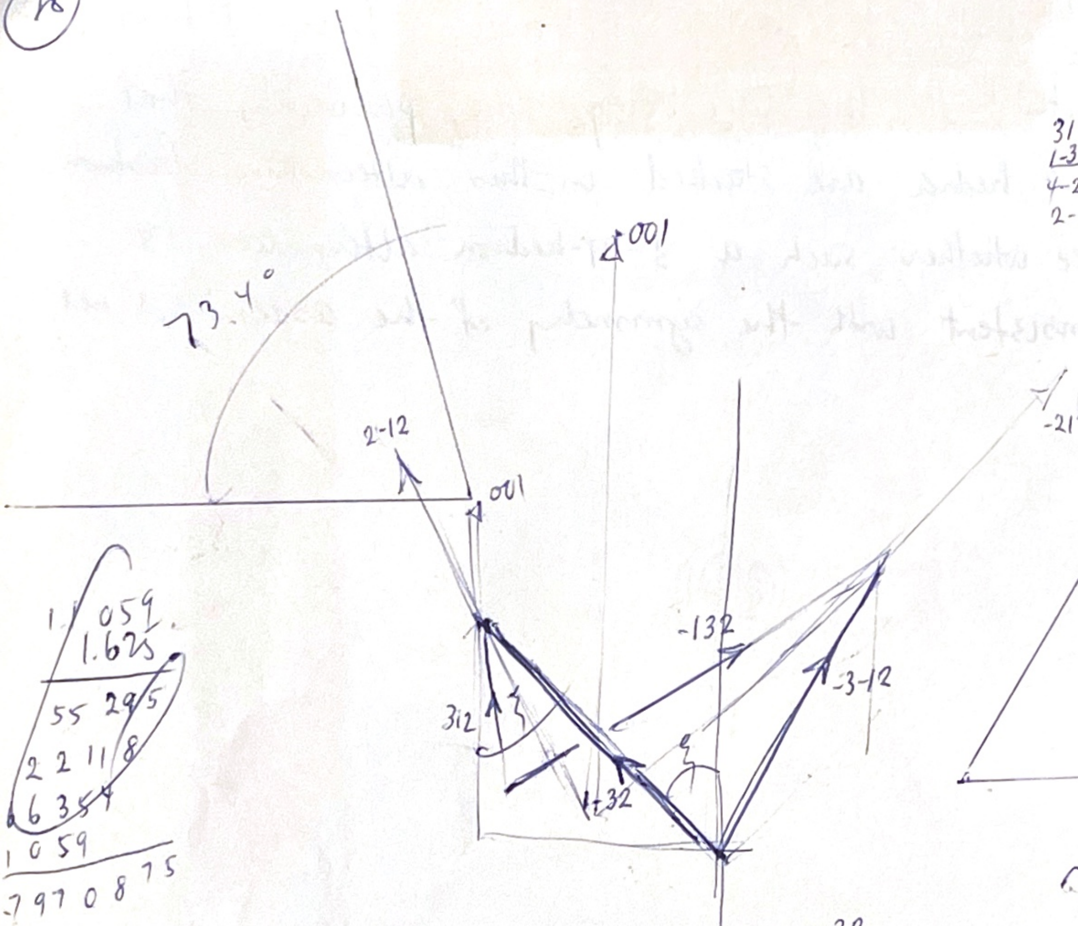
The symmetry domain? (If so, it's at least a good deal more interesting than the guess depicted 2 pp. back.)

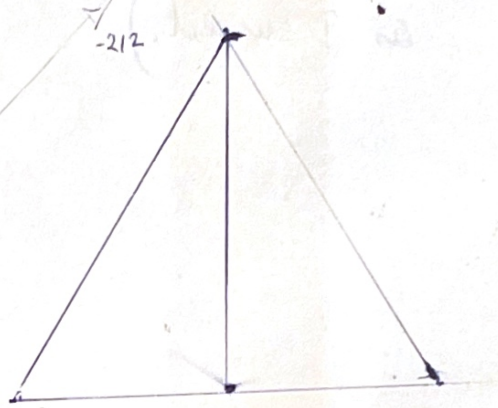
of one another, with a common (111) axis. Thus, I have
arbitrarily rejected, for the time being, the possibility that trihedra and 9-hedra are stacked in this alternating fashion. (I'll have to see whether such a 3-9-hedron alternation is in fact inconsistent with the symmetry of the assembled net, as I suspect.)

312 / $\underline{1\bar{3}2}$ / $4\bar{2}4$ / $2\bar{1}2$

$\bar{3}12$ / $\underline{\bar{1}32}$ / $\bar{4}24$ / $\bar{2}12$

$\bar{2}12$ / $\underline{2\bar{1}2}$ / 004





1.1059
1.625
55295
22118
66354
11059
1.7970875

1.625
.63258
13000
8125
3250
4875
1750
.02794250

?!!

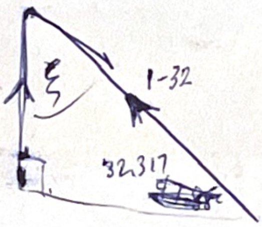
$\dfrac{\begin{matrix}1\bar{3}2\\ \cdot\,001\end{matrix}}{\dfrac{002}{\sqrt{14}}} = \dfrac{2\sqrt{14}}{14} = \dfrac{\sqrt{14}}{7} = \dfrac{\cancel{5.192}}{7} = \cancel{.7417}$

$\dfrac{3.742}{7} = .5346$

$\xi = \cos^{-1}(\overset{.5346}{\cancel{.7417}}) = 57.683°$

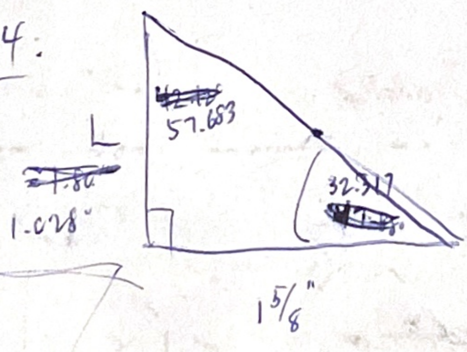
$= 90° - \overset{32.317}{\cancel{47.88}} = \cancel{42.12}$

$\hat{n} \propto (001)$



$\xi = \overset{57.683°}{\cancel{42.12°}}$

Make 4:

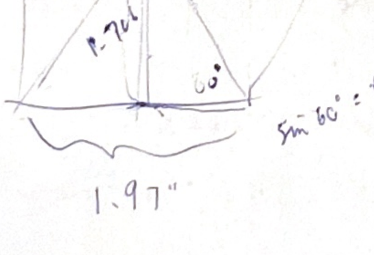


$1.625 \tan(\overset{32.317}{\cancel{47.88°}})$

$= 1.625\,(\overset{.63258}{\cancel{1.1059}}) = \cancel{1.797 \cong 1.80}$

1.028"

1.97



$\sin 60° = \frac{\sqrt{3}}{2} = .866$ / $\underline{197}$

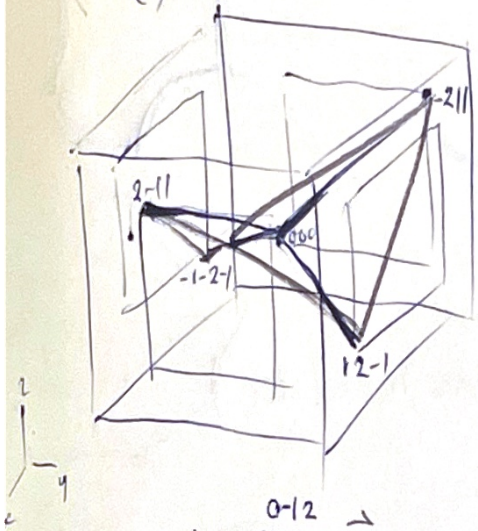
1.732
-.06
.10392
2598

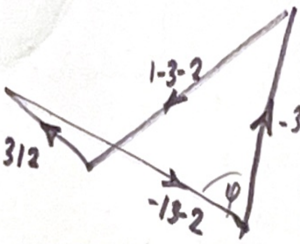
1.732
~~.104~~
1.628

1.732
-.026
1.706

The $\{6, 4\}_{112}$ inf. reg. sad. polyhedron
leads to two inf. uniform saddle polyhedra (quasi-regular polyhedra)

(1) Form vertex figures at alternate vertices. This leads to the (123) 8-connected homog. isotropic net (or rather its edges). Now ~~span~~ the (123) quadrilaterals and the plane triangles which are defined in every one of the original (112) = regular skew hexagons.



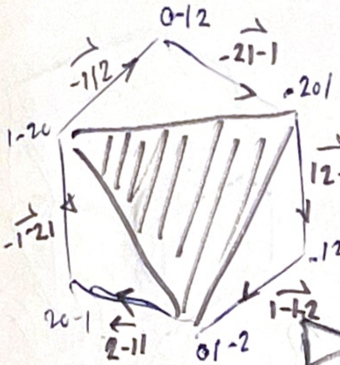


$\varphi = \cos^{-1}\left(\frac{2}{7} = .285714\ldots\right)$

$7\overline{)2.0000}$ = .285714

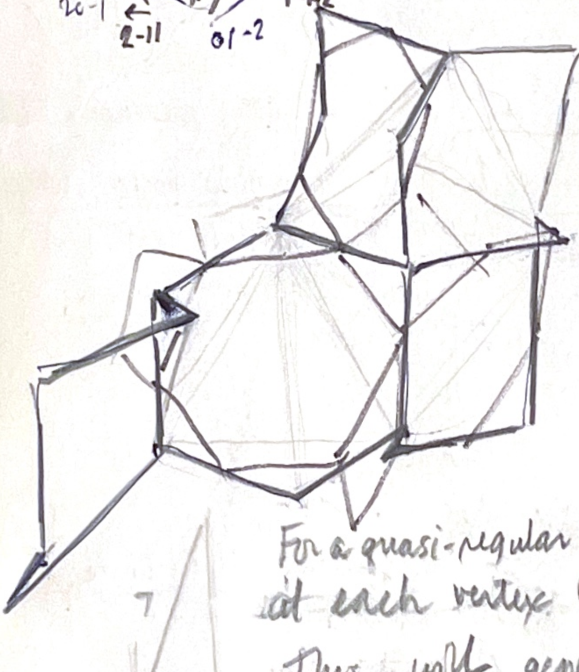
$90° - 16.602 = \boxed{73.398°}$

$\frac{-3\ 1\ 2 \cdot 1\ 3\ 2 = -3+3+4}{14} = \frac{2}{7}$

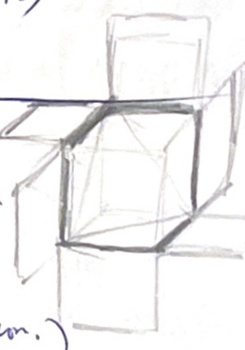


(2) Form the "$\frac{1}{2}$" vertex figures, i.e., the figures geometrically similar to, but one-half the linear dimensions of, the vertex figure (generated by joining the mid-points of ~~the~~ adjacent ~~adj~~ edges meeting at a single vertex).

This leaves plane hexagons, instead of triangles, in the original (112) regular skew hexagon faces.



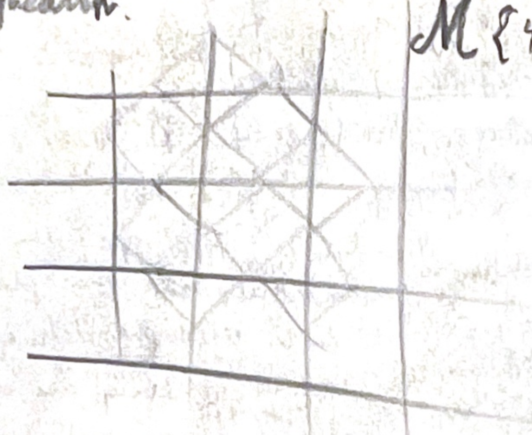
(The $\{4, 6\}_{111}$ ∞ reg. sad. polyhedron leads to only 1 uniform figure, the 90° skew hex & square polyhedron.)

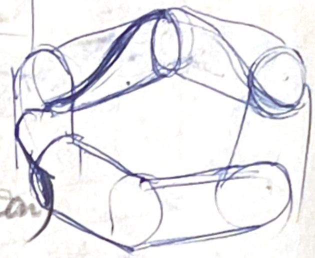


For a quasi-regular polyhedron, join midpoints of edges meeting at each vertex (i.e., construct vertex figures).
This will generate a uniform 4-connected net.
Of course, the vertex figures of quasi-regular polyhedra are not regular polygons. They are cyclic and equiangular, in the case of convex polyhedra. In the case of the skew polyhedra, saddle polyhedra, and skew saddle polyhedra, the vertex figures can be inscribed in a sphere (because the edges at each vertex, at any stage of mid-point connecting, are of equal length).

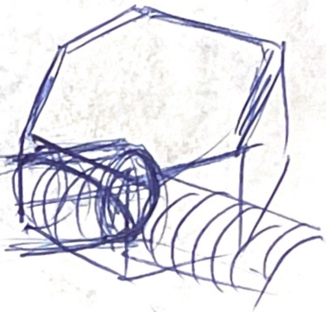
(30) Let $\mathcal{M}$ ≡ operator which constructs vertex figures of all the faces of a polyhedron.

$$\mathcal{M}\{4,4\} = \{4,4\} = \begin{Bmatrix}4\\4\end{Bmatrix}$$



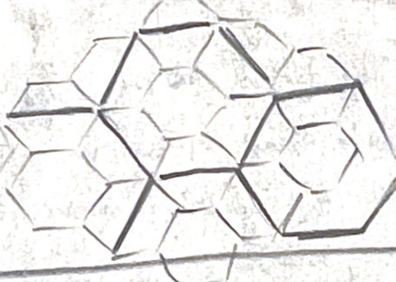


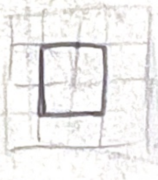
$$\mathcal{M}\{3,6\} = \begin{Bmatrix}3\\6\end{Bmatrix}$$ (quasi-regular tessellation)



$$\mathcal{M}\{6,3\} = \begin{Bmatrix}6\\3\end{Bmatrix} \left(= \begin{Bmatrix}3\\6\end{Bmatrix}\right)$$

Let $\mathcal{Q}$ ≡ operator which joins the vertices of alternate edges meeting at a given vertex, or — more generally — joins each vertex to every vertex which is the next closest, after the vertices of consecutive edges.





Let $\mathcal{N}$ ≡ operator which joins vertices of consecutive edges meeting at alternate vertices

(Doesn't apply here)