Monday, April 1, 1968

I believe that I have discovered the underlying net of "asymptotics" (Cf. H A Schwarz) for the ∞ per. minimal surface separating the two "Laves" labyrinths (enantiomorphous). The asymptotics are infinite (L.H. & R.H.) helices. They replace the straight edges of the net underlying $\{\tilde{6}, \tilde{4}\}_{Laves}$. As in the case of the 3 Schwarz IPMS's $[\{\tilde{6}, 4\}, \{\tilde{4}, 6\}$, and $\{6, 6\}]$, I believe it will be found that the helices which intersect always do so at right angles. Furthermore, I suspect that this single IPMS provides the surface which can be "tessellated" into counterparts of any one of the three ∞ regular [skew saddle] polyhedra $\{\tilde{6}, \tilde{4}\}_{Laves}$, $\{\tilde{4}, \tilde{6}\}_{Laves}$, or $\{\tilde{6}, \tilde{6}\}_{Laves}$, in the special sense that the helical-edged polygons which correspond to the straight-edged polygons of these polyhedra will be found to lie in this surface.

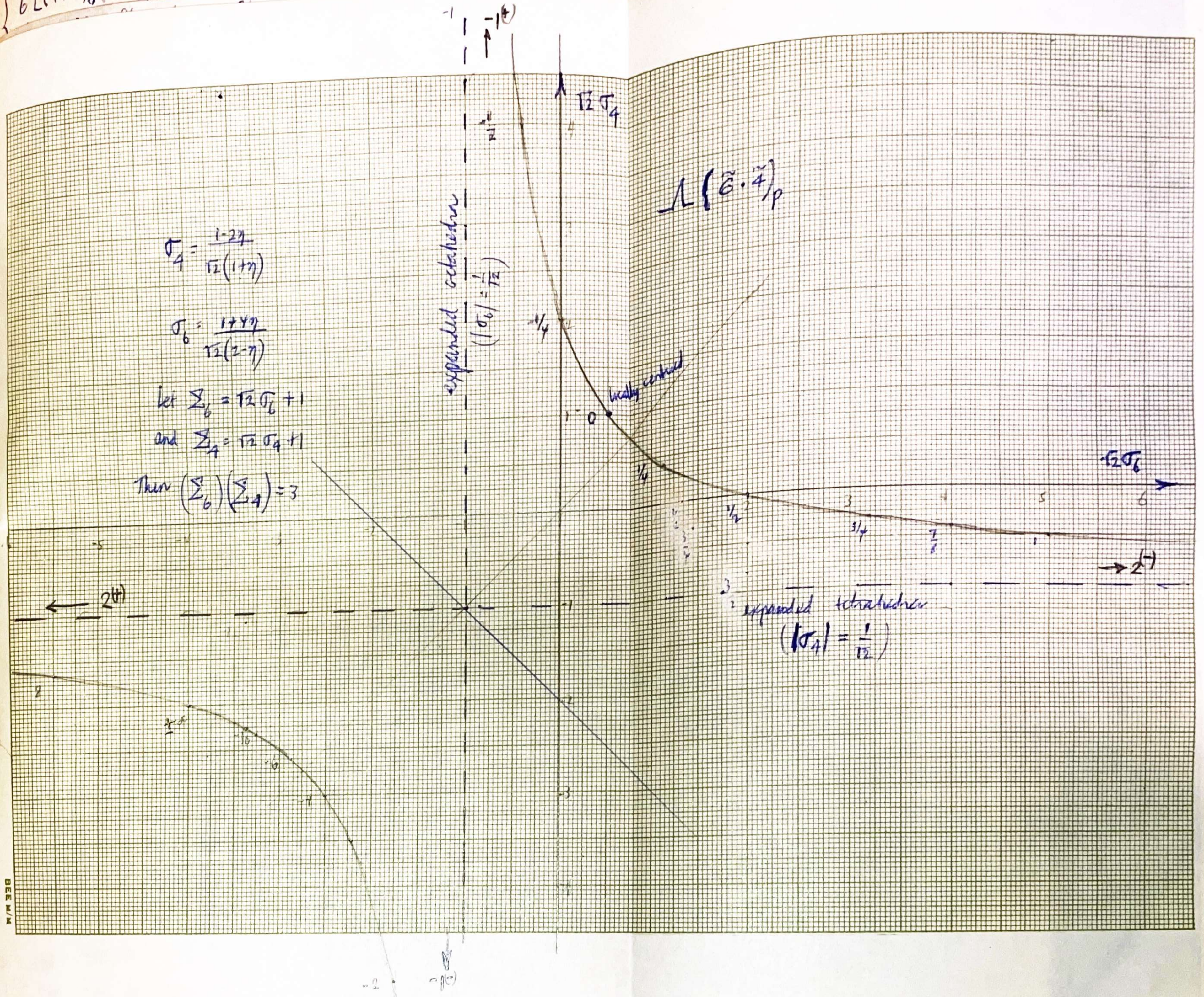
I suspect that the symmetry operation which generates the whole IPMS from a single module (this single module could be any one of the three counterpart helical-edged polygons of the modules of $\{\tilde{6}, \tilde{4}\}_{Laves}$, $\{\tilde{4}, \tilde{6}\}_{Laves}$, or $\{\tilde{6}, \tilde{6}\}_{Laves}$) is the screw operation. For example, let $\{\tilde{6}, \tilde{4}\}'_{Laves}$ denote the helical-edged tessellation corresponding to $\{\tilde{6}, \tilde{4}\}_{L}$. Then the six "polygons" which share (helical) edges with any one are generated by a 90° screw operation (alternately L.H. or R.H.), going around the edges of a single module).

The fact that $\{\tilde{6},\tilde{4}\}_L$ and $\{\tilde{4},\tilde{6}\}$ have 4-fold screw-symmetric "holes" (Cf. Coxeter; Coxeter & Moser) which are regular helical polygons ($\{4_h\}$), while $\{\tilde{6},\tilde{6}\}_L$ has 3-fold screw-symmetric "holes" suggests that in the Laves graph ∞ periodic minimal surface (S_L) there are 3 sets of ~~orthog~~ intersecting helical "asymptotics". Perhaps all three sets intersect ~~asym~~ orthogonally.

The "invariant points" which lead to the definition of S_L are those points common to $\{\tilde{6},\tilde{4}\}_L$, $\{\tilde{4},\tilde{6}\}_L$, and $\{\tilde{6},\tilde{6}\}$. ~~which~~ These points are points at which the point group operators of the [common] space group lie. These points lie at the lattice points of each of the symmetrically superimposed space lattices (b.c.c.) of each homogeneous isotropic net appropriate to a particular one of the Laves polyhedra. These invariant points define S_L in the sense that they lie in S_L, and the helical asymptotics of S_L are geodesics which join adjacent invariant points.

Each helical-edged polygon is a regular h-e-p., because there is a symmetry operator (roto-reflection) which leaves the h.e.p. invariant.

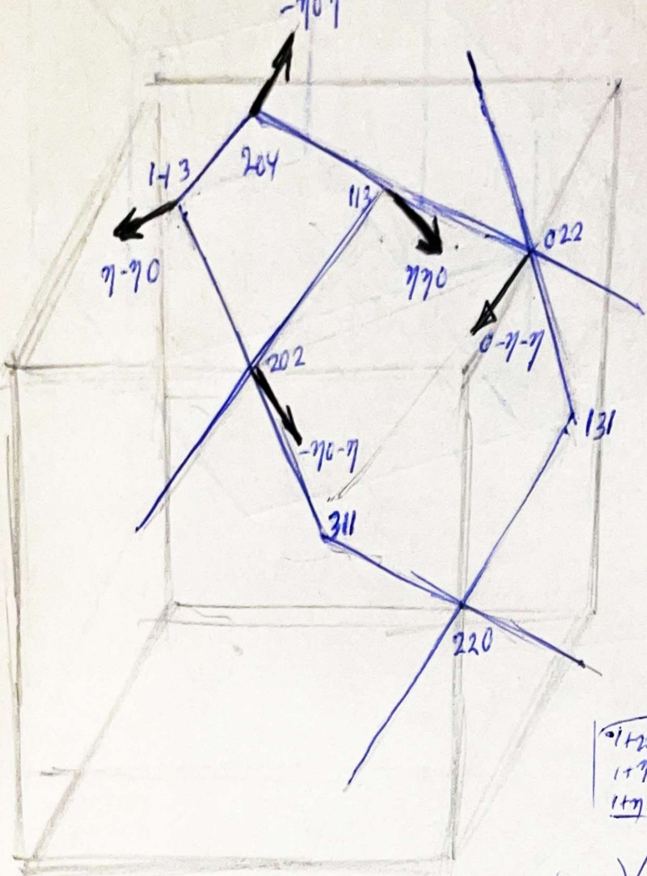
$\{\tilde{6}[(1+4\eta)/(\sqrt{2}(2-\eta))]\}$ (locally centred continuation of

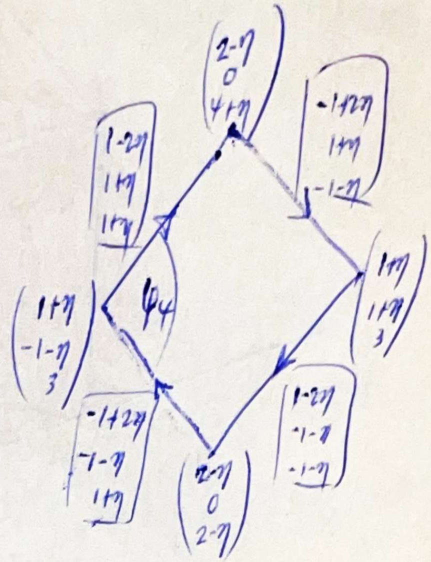


6 L(…) (locally centred configuration at …

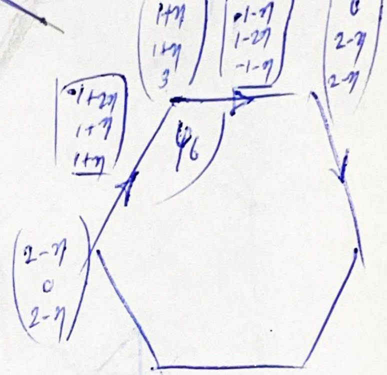
	$7/2$	$9/2$	$9/8$	4	$7/4$	$-3/4$	$15/8$	$-.4$	
				σ_6				σ_4	
η	4η	$1+4\eta$	$2-\eta$	$\frac{1+4\eta}{2-\eta}$	2η	$1-2\eta$	$1+\eta$	$\frac{1-2\eta}{1+\eta}$	
$-\infty$	$-\infty$	$+\infty$	∞	-4	$-\infty$	∞	$-\infty$	-2	
-4	-16	-15	6	$-\frac{5}{2}$	-8	9	-3	-3	
-2	-8	-7	4	$-7/4$	-4	5	-1	-5	
-1	-4	-3	3	-1	-2	3	0	∞	
$-1/2$	-2	-1	$5/2$	$-\frac{2}{5}$	-1	2	$\frac{1}{2}$	4	
$-1/4$	-1	0	$9/4$	0	$-1/2$	$3/2$	$\frac{3}{4}$	2	
0	0	1	2	$1/2$	0	1	1	1	
$1/4$	1	2	$7/4$	$\frac{8}{7}=1.143$	$1/2$	$1/2$	$5/4$	$\frac{2}{5}$	
$1/2$	2	3	$3/2$	2	1	0	$3/2$	0	
1	4	5	1	5	2	-1	2	$-1/2$	
2	8	9	0	$\pm\infty$	4	-3	3	-1	
4	16	17	-2	$-\frac{17}{2}$	8	-7	5	$-\frac{7}{5}$	
8	32	33	-6	-5.5	16	-15	9	$-5/3=-1.067$	
∞	∞	∞	$-\infty$	-4	∞	$-\infty$	∞	-2	
-6	-24	-23	8	$-\frac{23}{8}=-2.875$	-12	13	-5	$-\frac{13}{5}$	$=-2.6$
-3	-12	-11	5	$-\frac{11}{5}=-2.2$	-6	7	-2	$-7/2$	$=-3.5$
$\frac{1}{8}$	$1/2$	$3/2$	$\frac{17}{18}$	$\frac{27}{17}=1.588$	$1/4$	$3/4$	$9/8$	$2/3$	$.666$
$\frac{3}{8}$	$3/2$	$5/2$	$\frac{19}{8}$	$\frac{20}{19}=1.052$	$\frac{3}{4}$	$1/4$	$11/8$	$\frac{2}{11}$	$.1818$
$\frac{3}{4}$	3	4	$\frac{5}{4}$	$\frac{16}{5}=\cancel{3.2}$	$\frac{3}{2}$	$-1/2$	$7/4$	$-\frac{2}{7}$	$-.2857$
$3/2$	6	7	$\frac{1}{2}$	$\frac{\cancel{2}}{14}$	3	-2	$5/2$	$-4/5$	$-.8$
-12	-48	-47	14	-3.357	-24	25	-11	$-\frac{25}{11}$	$=\cancel{-1.087}$ -2.273
-100	-400	-399	102	$\frac{-399}{102}=-3$					
-10	-40	-39	12	-3.25	-20	21	-9	$-\frac{7}{3}$	$=-2.333$

$\left\{\begin{array}{l}\tilde{6}\,[(1+4\eta)/\sqrt{2}(2-\eta)]\\ \tilde{4}\,[(1-2\eta)/\sqrt{2}(1+\eta)]\end{array}\right\}$ (locally centred configuration of $(\tilde{6}\cdot\tilde{4})_p$)



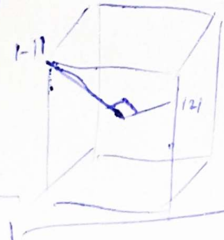


$$\cos\varphi_4 = \frac{\begin{pmatrix}1-2\eta\\1+\eta\\1+\eta\end{pmatrix}\begin{pmatrix}1-2\eta\\1+\eta\\-1-\eta\end{pmatrix}}{3+6\eta^2} = \frac{(1-2\eta)^2}{3+6\eta^2}$$



$$\cos\varphi_6 = \frac{\begin{pmatrix}1+\eta\\-1+2\eta\\1+\eta\end{pmatrix}\begin{pmatrix}-1+2\eta\\1+\eta\\1+\eta\end{pmatrix}}{3+6\eta^2} = \frac{-1+4\eta+5\eta^2}{3+6\eta^2}$$

$$\boxed{\begin{array}{l}\sigma_4 = \dfrac{1-2\eta}{\sqrt{2}(1+\eta)}\\[2ex] \sigma_6 = \dfrac{1+4\eta}{\sqrt{2}(2-\eta)}\end{array}}$$

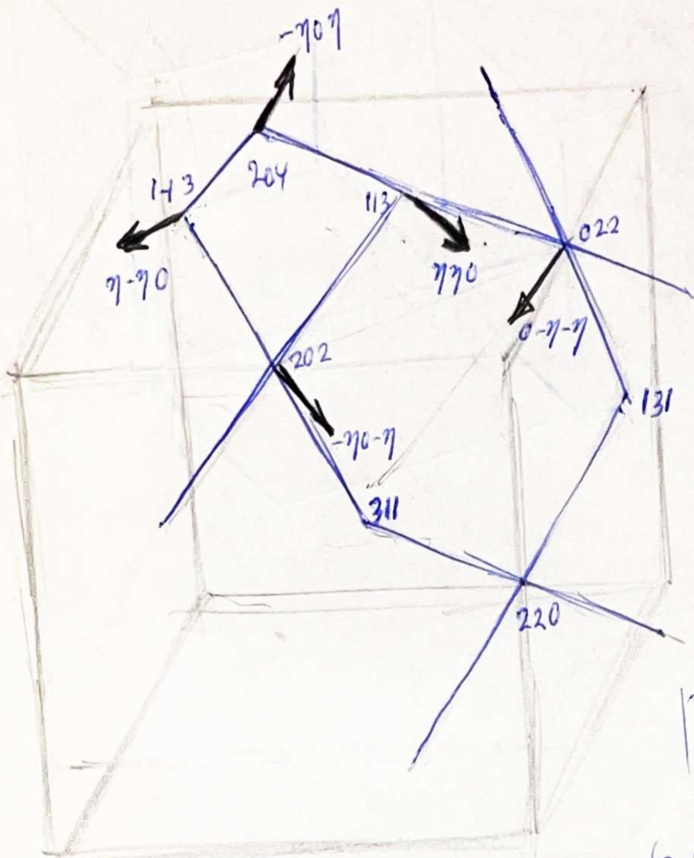


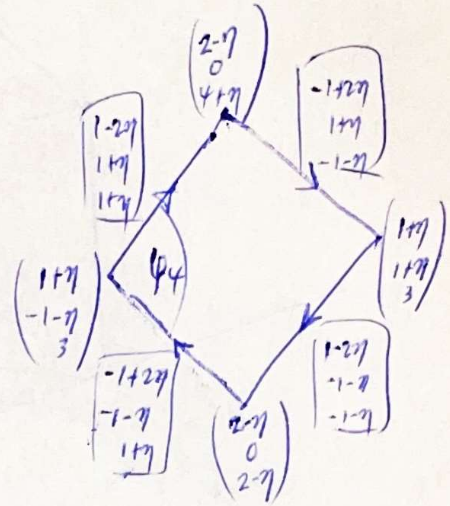
$\Delta\alpha = 90°$

$$\boxed{\frac{l_0}{l} = \cos\alpha = \frac{1}{\sqrt{1+2\eta^2}}} \quad \boxed{\eta = \frac{\sqrt{2}}{2}\tan\alpha}$$

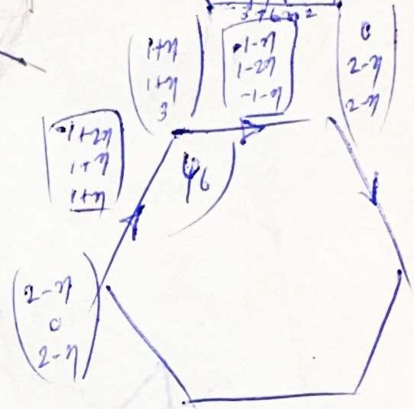
$|l| = \sqrt{3}$

$$\left.\begin{aligned}&\tilde{6}\,[(1+4\eta)/(\sqrt{2}(2-\eta))]\\&\tilde{4}\,[(1-2\eta)/\sqrt{2}(1+\eta)]\end{aligned}\right\}$$ (locally centred configuration of $(6\cdot4)_p$)





$$\cos\varphi_4 = \frac{\begin{pmatrix}1-2\eta\\1+\eta\\1+\eta\end{pmatrix}\cdot\begin{pmatrix}1-2\eta\\1+\eta\\-1-\eta\end{pmatrix}}{3+6\eta^2} = \frac{(1-2\eta)^2}{3+6\eta^2}$$



$$\cos\varphi_6 = \frac{\begin{pmatrix}1+\eta\\-1+2\eta\\1+\eta\end{pmatrix}\cdot\begin{pmatrix}-1+2\eta\\1+\eta\\1+\eta\end{pmatrix}}{3+6\eta^2} = \frac{-1+4\eta+5\eta^2}{3+6\eta^2}$$

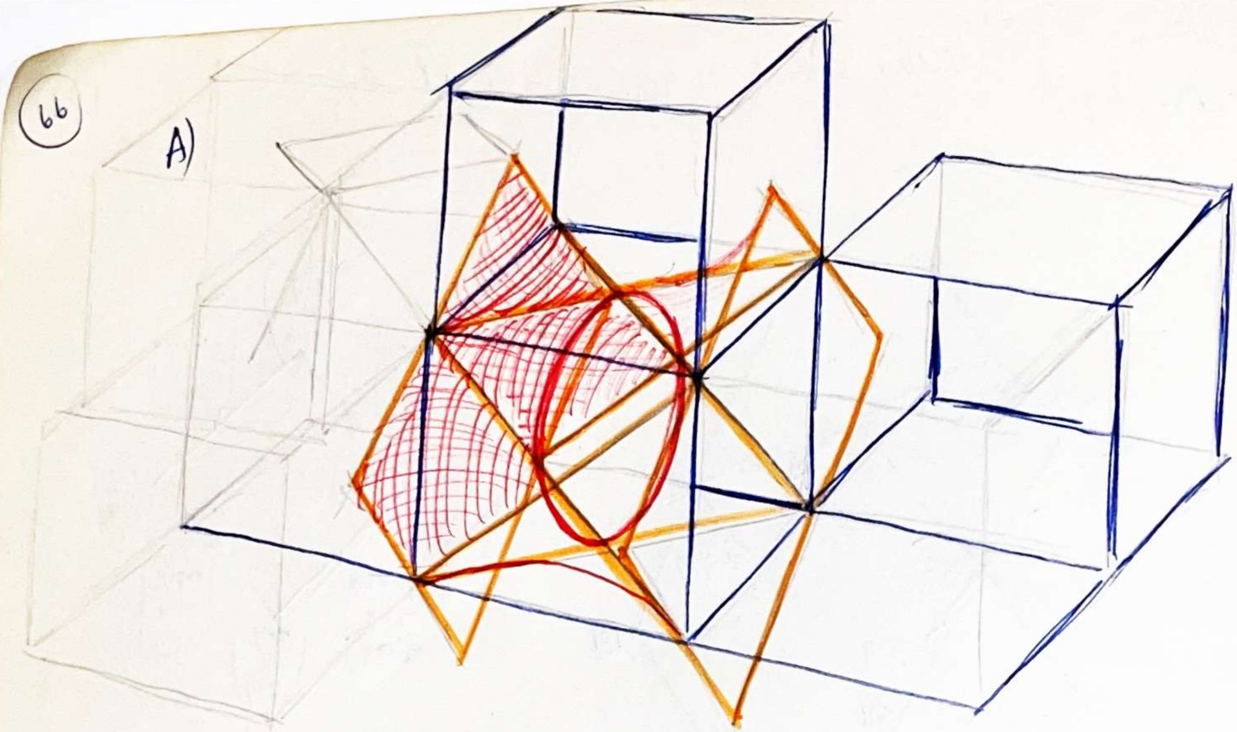
$$\boxed{\begin{aligned}\sigma_4 &= \frac{1-2\eta}{\sqrt{2}(1+\eta)}\\[1em] \sigma_6 &= \frac{1+4\eta}{\sqrt{2}(2-\eta)}\end{aligned}}$$

$\Delta\alpha = 90°$

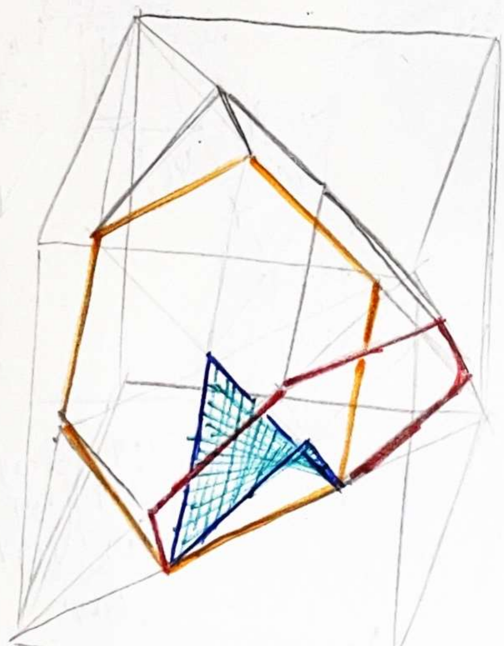
$$\boxed{\frac{l_0}{l} = \cos\alpha = \frac{1}{\sqrt{1+2\eta^2}}} \qquad \boxed{\eta = \frac{\sqrt{2}}{2}\tan\alpha}$$

$|l| = \sqrt{3}$

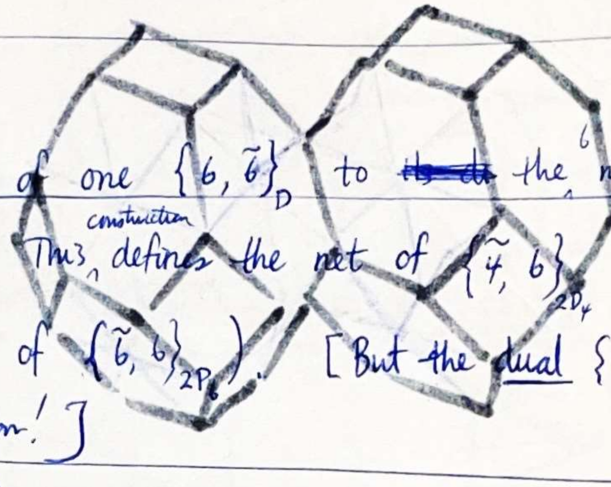
A)



B)



Let us describe a certain construction which transforms the dual pairs of regular skew polyhedra into the corresponding Schwarz minimal surfaces.



A) Diamond: $2D_4$

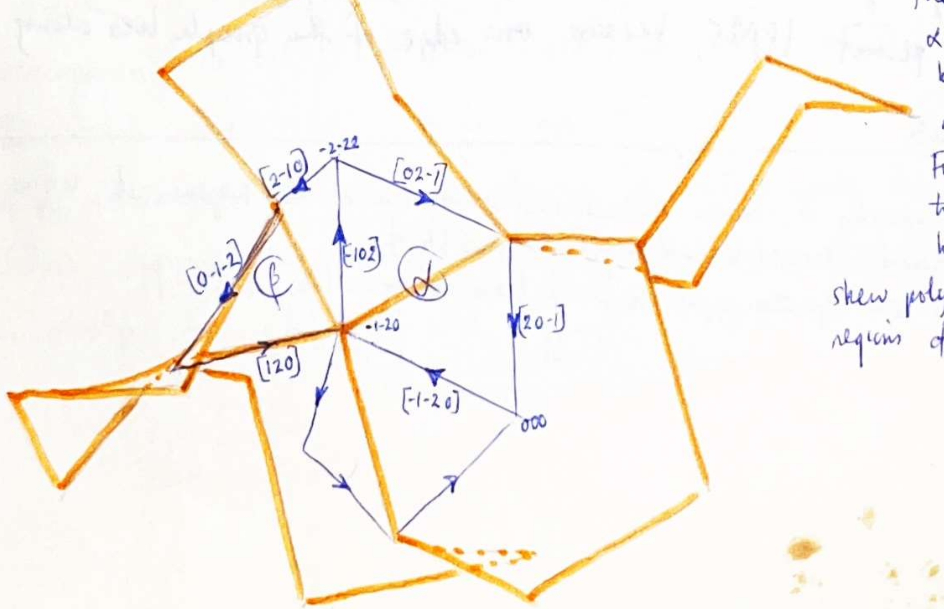
Connect each of the vertices of one $\{6,\tilde{6}\}_D$ to ~~the~~ the 6 nearest vertices of its dual $\{6,\tilde{6}\}'_D$. This construction defines the net of $\{\tilde{4},6\}_{2D_4}$ (and therefore also — of course — of $\{\tilde{6},6\}_{2P_6}$). [But the dual $\{\tilde{6},6\}_{2P6}$ net is not developed by this construction!]

B) Simple cubic: $2P_6$

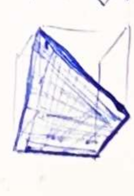
Connect each vertex of $\{6,\tilde{4}\}_P$ to the 4 nearest vertices of its dual $\{4,\tilde{6}\}_P$. This construction defines the superposition of the net of $\{\tilde{6},6\}_{2P_6}$ and its dual $\{\tilde{6},6\}'_{2P_6}$.

I have been unable to find any constructions analogous to this one for developing the asymptotics of $2L_3$. If one carries out exactly the construction described above, the results are as follows:

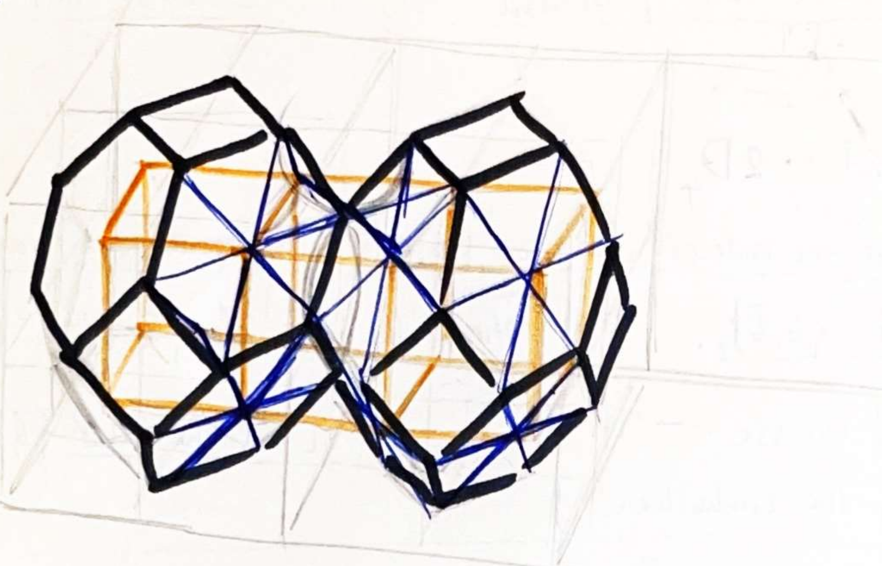
Consider the dual polyhedra $\{\tilde{4},\tilde{6}\}_L$ and $\{\tilde{6},\tilde{4}\}$



The quadrilaterals α & β are not related by rotation through π about their common edge. Furthermore, the skewness transformation does not help. (It deforms the skew polygons into the fundamental regions of $\{\tilde{6},4\}_\delta$, when $\eta = 1$.)

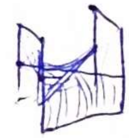


A

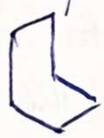


So far, it appears that every non-self-intersecting IPMS is based on labyrinths whose skeletons are topologically equivalent to one of the following 3 graphs: P, D, or L. For example, a) rhombohedral graphite;

b) the 4-connected labyrinth of the Schwarz IPMS;



FALSE: Consider the tetragonal case on p. 74!

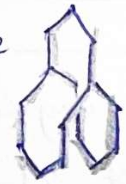


It is based on this hexagon.

The nets of the labyrinths are 2 interpenetrating nets of this type

(same net, topologically)

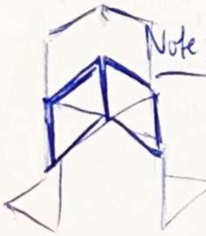
c) the



Bucket-handle net of Wells;

and d) kagome (tetrahedra — truncated tetrahedra interstitial domains)

are all self-interstitial nets. Their labyrinths are topologically equivalent to diamond. It's not clear that the symmetry of a) and c) permit IPMS, because one edge of the graph lies along the preferred axis.



Note: An assembly of rhombic dodecahedra on the sites of the kagome net defines a rhombic-faceted labyrinth modification of $\{\tilde{6},4\}_D$

The same operation applied to $\{\tilde{6},6\}$ leads to square faces, & $\{4,\tilde{6}\}$.

$\{\tilde{6},\tilde{6}\}$ " " $\{\tilde{4},\tilde{6}\}$

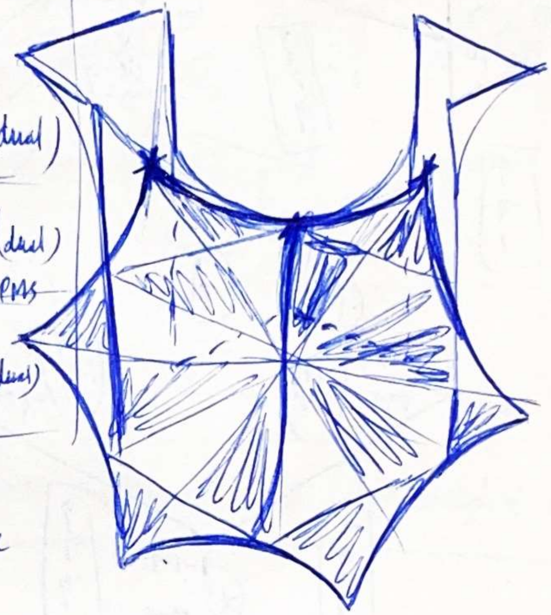
April 11, 1968

More important — more fundamental — is the fact that all IPMS — whether on a cubic lattice or not, correspond to labyrinth graphs which are related to each other by 2-fold symmetry operators of their superposed configuration.

The one property common to all IPMS on a cubic space lattice, with no self-intersections, is that the skeletons of their labyrinths define self-interstitial nets: Thus, the symmetry domain and interstitial domain of the underlying net are congruent, and each is centered on the nodes of the net of the other.

The net formed by joining alternate vertices of $(10)_3$ has this property. Its edges can be curved to form circles to make a particularly symmetric form. Does this lead to 3 regular polyhedra and an IPMS? (How can I find out?!)

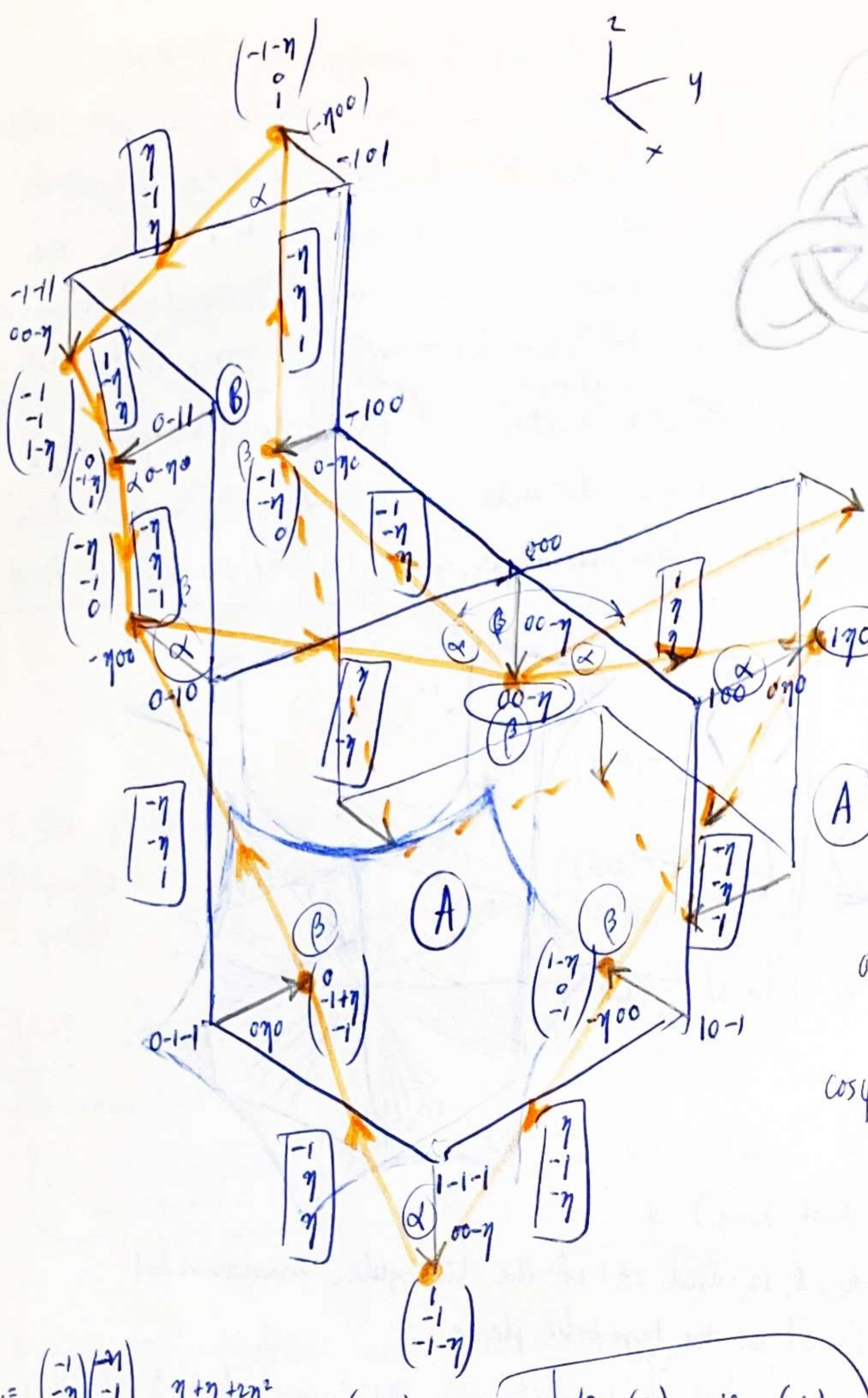
D	◇	$\{\tilde{4}, 6\} \leftrightarrow \{\tilde{6}, 4\}$ IPMS IPMS	$\{6,6\}$ $\langle \tilde{6}, \tilde{6} \rangle \leftrightarrow$ (dual)
P	SC	$\langle \tilde{4}, \tilde{6} \rangle \leftrightarrow \langle \tilde{6}, \tilde{4} \rangle$	$\{\tilde{6}, 6\} \leftrightarrow$ (dual) IPMS IPMS
L	L	$\{\tilde{4}, \tilde{6}\} \leftrightarrow \{\tilde{6}, \tilde{4}\}$	$\{\tilde{6}, \tilde{6}\} \to$ (dual)

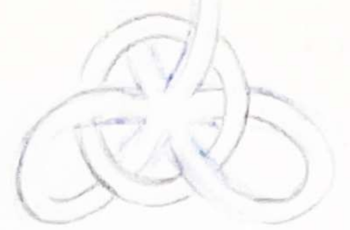


The 3 kinds (actually 4, counting the dual of $\{6,6\}$ as a distinct kind) of tessellations correspond to 8, 12, and 24 of the triangular fundamental regions of the group $[4,6]$ in the hyperbolic plane.

For D, P, and L, it is not possible to tessellate the IPMS into straight-edged modules corresponding to all 4 of these tessellations. Even in D & P, half of the tessellations have curved sides. In L, all 4 of them do.

It will be interesting to compare the contours of elastica with the sides of $\{6,6\}$ shown above. Doing this will require using the exact solutions of Schwarz for the Plateau problems for $\{\tilde{6}[1/2]\}$ and $\{\tilde{6}[\ ]\}$.





A $\cos\varphi_\alpha = \dfrac{\begin{pmatrix}-1\\ \eta\\ \eta\end{pmatrix}\begin{pmatrix}-\eta\\ 1\\ \eta\end{pmatrix}}{1+2\eta^2} = \dfrac{\eta+\eta+\eta^2}{1+2\eta^2} = \dfrac{\eta^2+2\eta}{1+2\eta^2}$

or $\boxed{\cos\varphi_\alpha = \dfrac{\eta(\eta+2)}{1+2\eta^2}}$

$\cos\varphi_\beta = \dfrac{\begin{pmatrix}\eta\\ -1\\ -\eta\end{pmatrix}\begin{pmatrix}\eta\\ \eta\\ 1\end{pmatrix}}{1+2\eta^2} = \dfrac{\eta^2-\eta-\eta}{1+2\eta^2} = \boxed{\dfrac{\eta(\eta-2)}{1+2\eta^2}}$

Thus, $\underline{\varphi_\alpha \neq \varphi_\beta}$

B $\cos\varphi_\alpha = \dfrac{\begin{pmatrix}-1\\ -\eta\\ \eta\end{pmatrix}\begin{pmatrix}-\eta\\ -1\\ \eta\end{pmatrix}}{1+2\eta^2} = \dfrac{\eta+\eta+2\eta^2}{1+2\eta^2} = \dfrac{\eta(\eta+2)}{1+2\eta^2}$

$\cos\varphi_\beta = \dfrac{\begin{pmatrix}-\eta\\ \eta\\ 1\end{pmatrix}\begin{pmatrix}1\\ \eta\\ -\eta\end{pmatrix}}{1+2\eta^2} = \dfrac{-\eta+\eta^2-\eta}{1+2\eta^2} = \dfrac{\eta(\eta-2)}{1+2\eta^2}$

$\therefore\ \varphi_\alpha(B) = \varphi_\alpha(A)$

$\therefore\ \varphi_\beta(B) = \varphi_\beta(A)$

Thus, we have here a kind of <u>semi-regular</u> saddle polyhedron.
The faces are all congruent semi-regular skew polygons,
the edges are all equivalent, and the vertices are all equivalent.
(One labyrinth shrinks and the other one expands, here.)

April 20, 1968

I hope now to treat in detail a <u>second</u> kind of skewness transformation.
In this case, <u>vertex displacements all occur from one labyrinth into the second.</u> In the case of quasi-regular polyhedra with Δ faces, there is no symmetric skewing transformation on the vertices of Type A (alternate vertices are displaced into alternate labyrinths). However, there <u>is</u> a symmetric <u>Type B</u> ~~transformation~~ which leads <u>not</u> to a quasi-regular polyhedron but to ~~a uniform~~ one having more than 2 kinds of regular faces.

To begin with easier examples, let us first consider σ_2 (Type B) operating on $\{\tilde{6}, 4\}_D$ (◊ Schwarz IPMS).

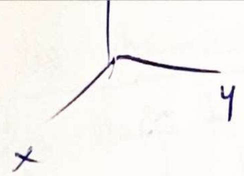
★ Call this (temporarily) $\sigma_2\{\tilde{6}, 4\}$, or $\Sigma_2\{\tilde{6}, 4\}$

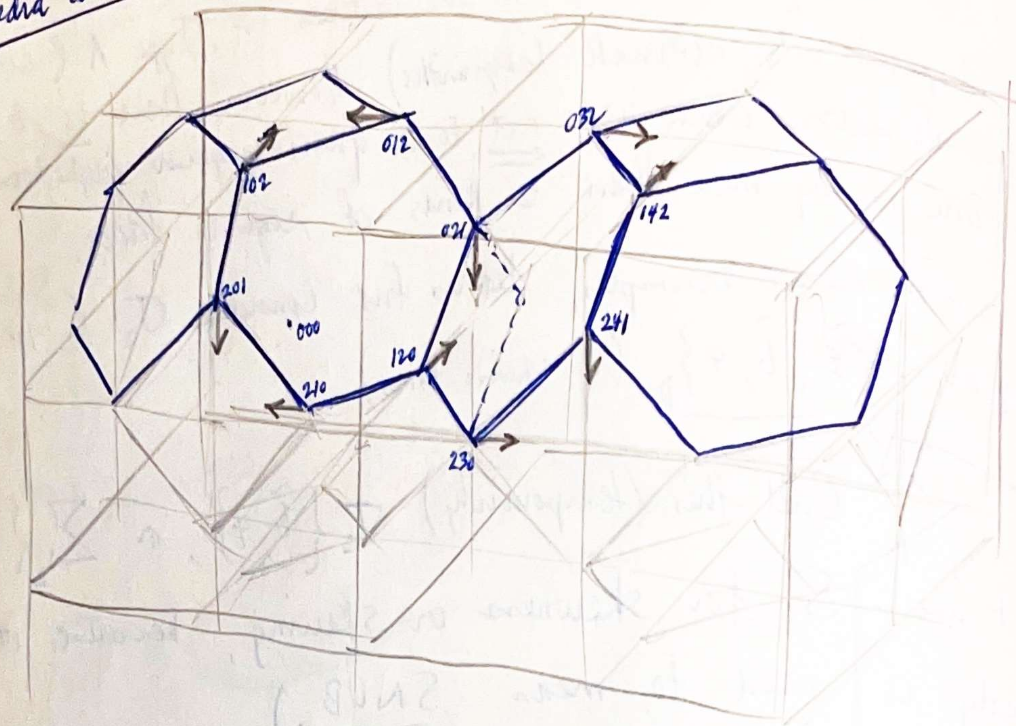
(Don't use S for skewness or skewing, because it already is used to mean <u>SNUB</u>.)

Try this next on $\{6, \tilde{4}\}_P$, the reg skew polyhedron.

WHAT do you get when you join adjacent vertices of dual Laves polyhedra ★?

Very interesting.
This collapses into a space-filling of octahedra and cuboctahedra!





Very interesting.
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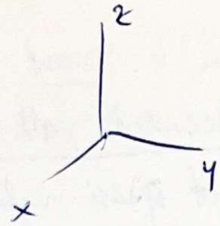
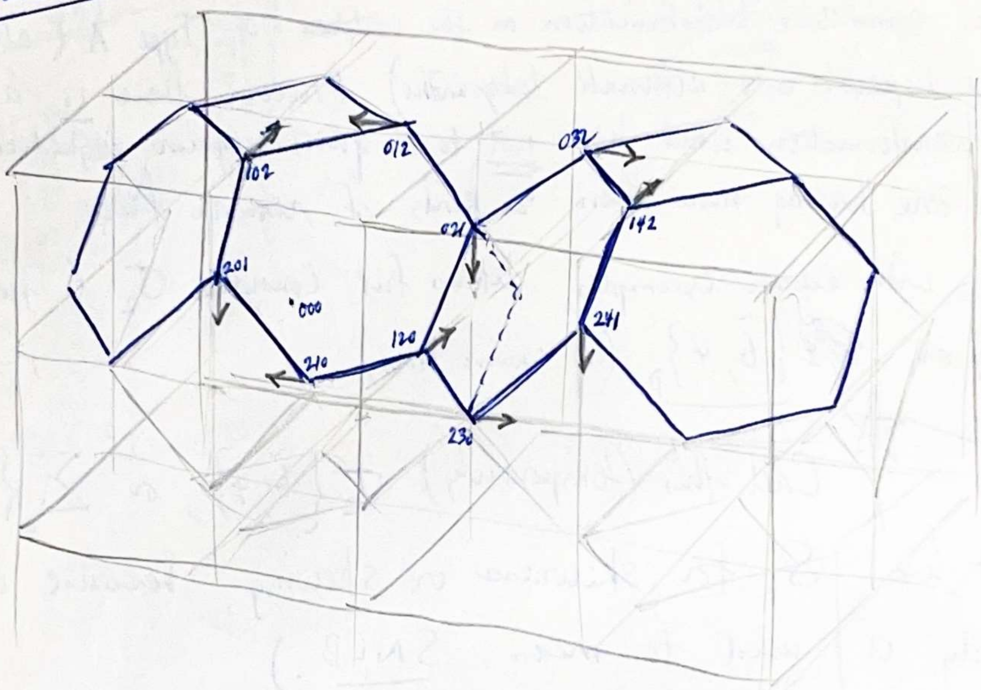
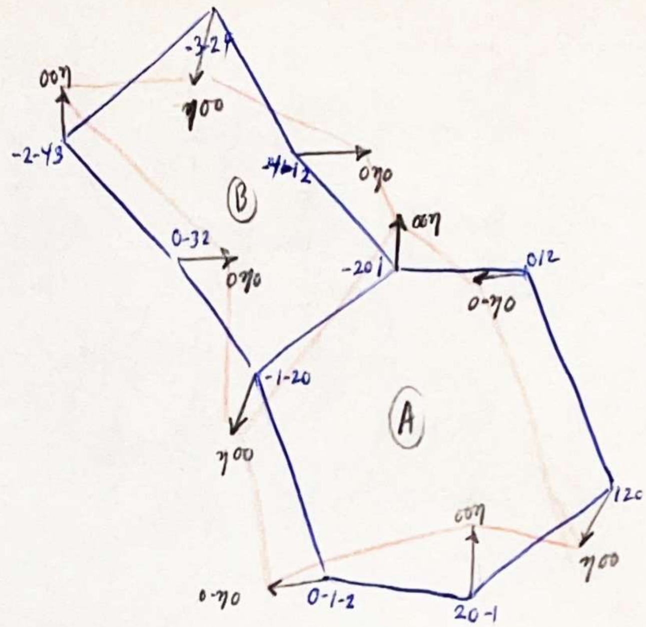
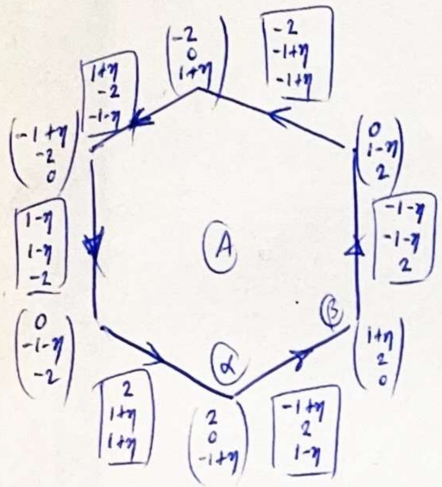


$\boxed{\sigma_2 [\{6, \tilde{4}\}]}$



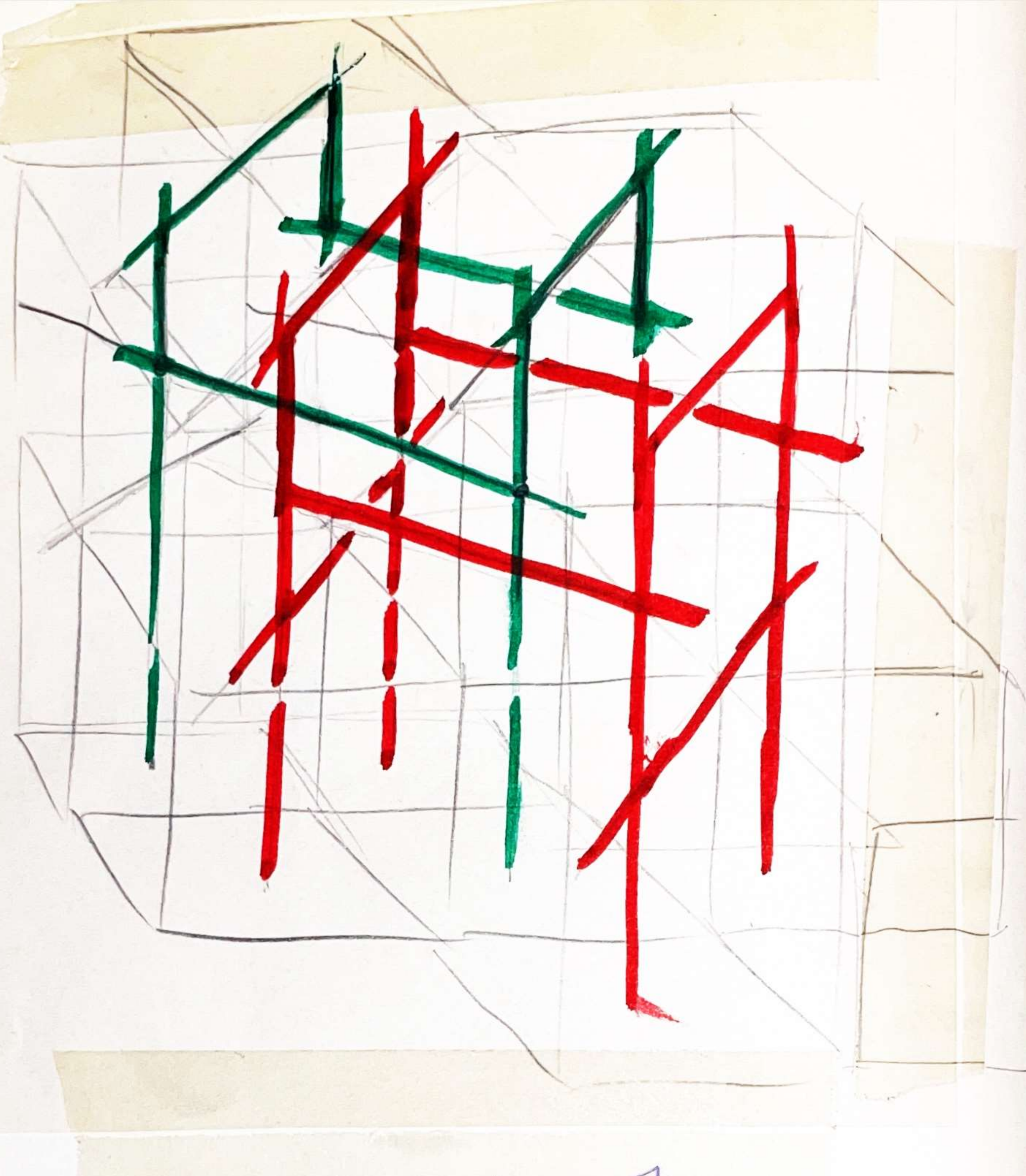




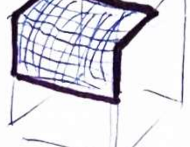
$$\cos\varphi_\alpha = \frac{\begin{pmatrix}-1+\eta\\2\\1-\eta\end{pmatrix}\begin{pmatrix}-2\\-1-\eta\\-1-\eta\end{pmatrix}}{\sqrt{2(1+\eta)^2+4}\sqrt{2(1-\eta)^2+4}} = \frac{2-2\eta+2+2\eta-(1-\eta^2)}{\sqrt{2+4\eta+2\eta^2+4}\sqrt{2-4\eta+2\eta^2+4}} = \frac{4-1+\eta^2}{\sqrt{2(\eta^2+2\eta+3)}\sqrt{2(\eta^2-2\eta+3)}}$$

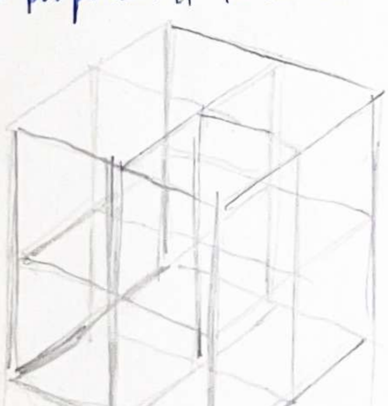
$$= \frac{3+\eta^2}{\sqrt{2(3+2\eta+\eta^2)}\sqrt{2(3-2\eta+\eta^2)}} = \boxed{\frac{3+\eta^2}{2\sqrt{(3+2\eta+\eta^2)(3-2\eta+\eta^2)}}}$$

$\cos\varphi_\beta =$

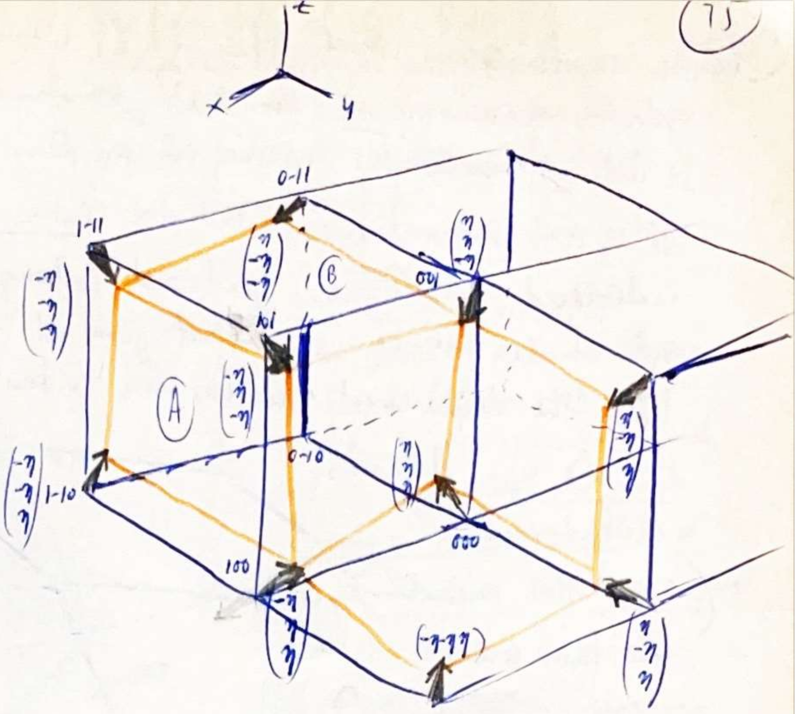


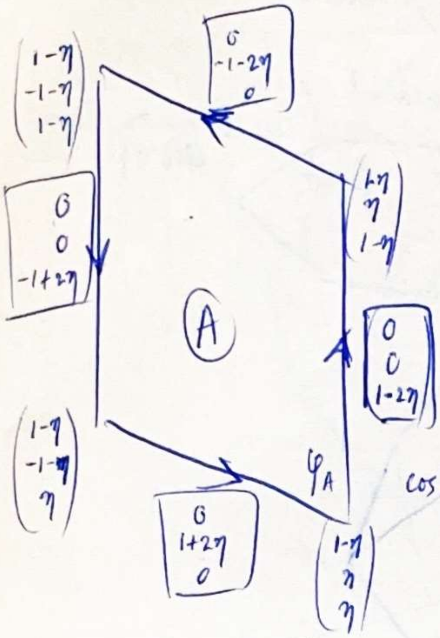
The nets of the labyrinths of the Schwarz IPMS based on 

If the proportions of the module are:  then the circuits of labyrinth net edges become:

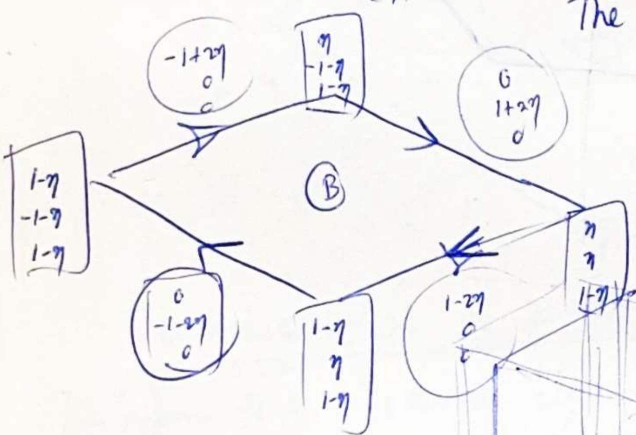


instead of:

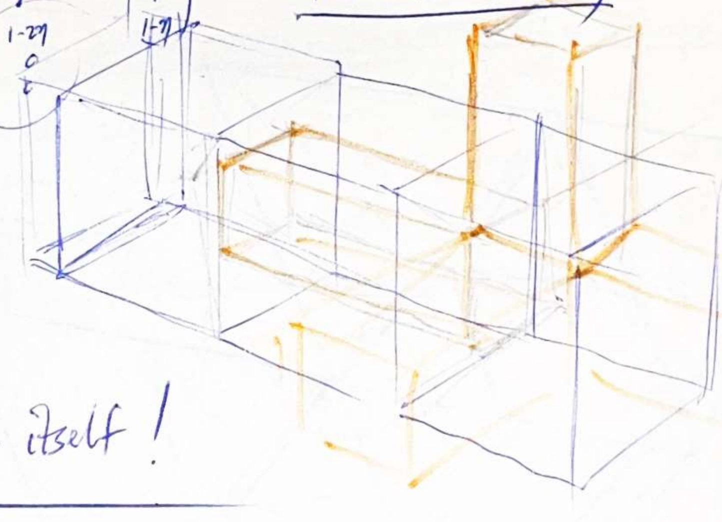




$$\cos\varphi_A = \frac{\begin{pmatrix}0\\0\\1-2\eta\end{pmatrix}\begin{pmatrix}0\\1+2\eta\\0\end{pmatrix}}{(1+2\eta)^2}$$



The angles remain 90° throughout
but the faces become rectangular
(Note that every vertex is proceeding toward the center of the "empty" cube, i.e., the face-less cube.)

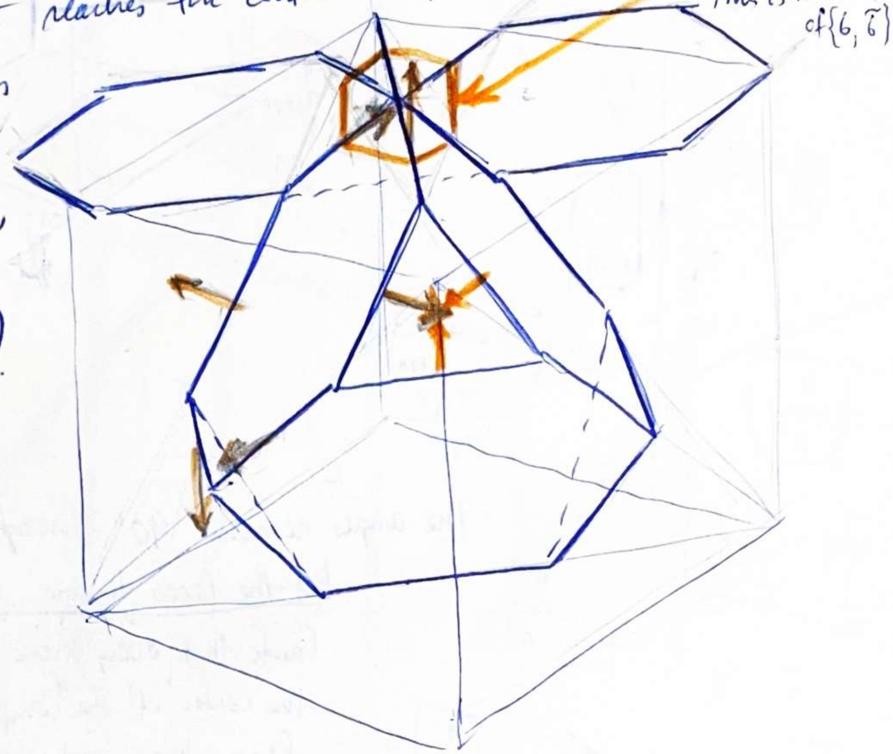


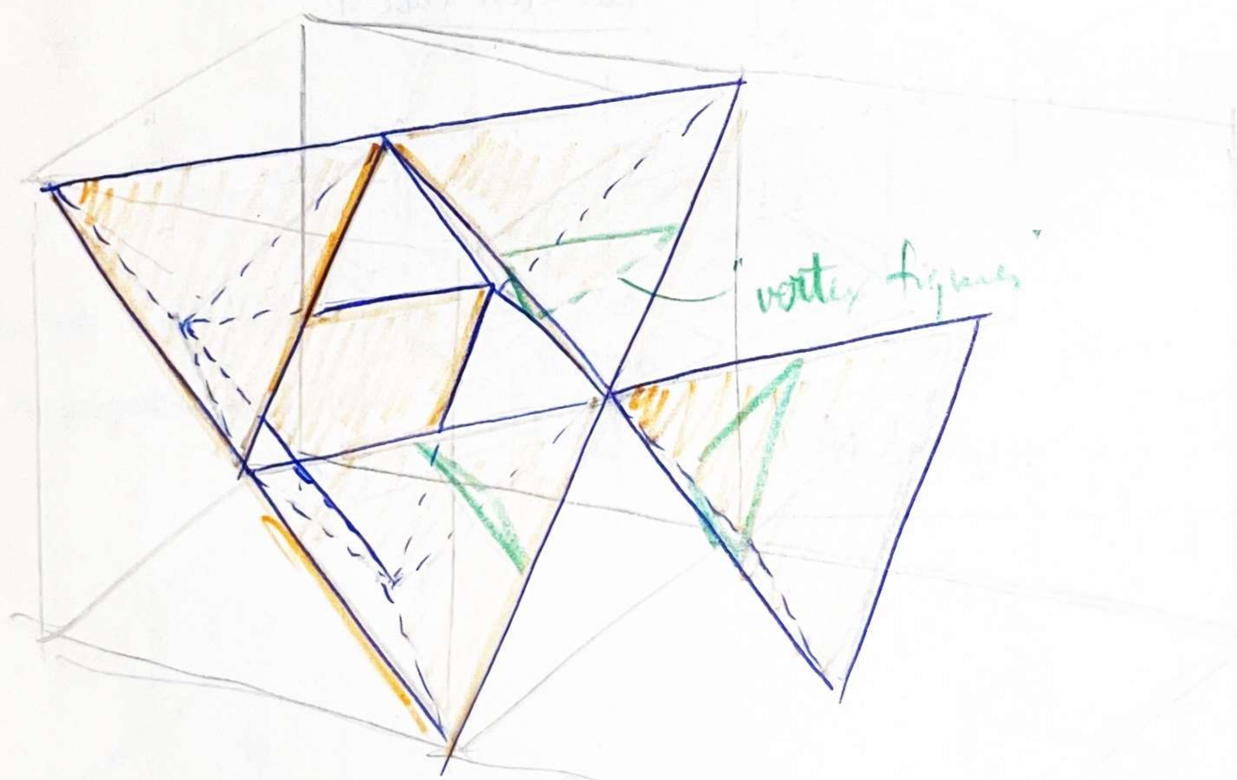
This eventually leads to the s.c. graph itself!

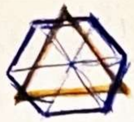
(76) The transformation which takes $\{6,\tilde{6}\}$ into a space filling assembly of diamond symmetry domains is not ~~a~~ skewing transformation applied at the vertices of $\{6,\tilde{6}\}$.
It does not preserve the regularity of the faces of $\{6,\tilde{6}\}$.

It is a transformation in which the center of each edge of $\{6,\tilde{6}\}$ is displaced symmetrically outward (along a cube axis) from the convex side of the pair of faces that join at that edge.
When the displacement reaches the center of the tetrahedral cavity, the ◊ sym. domains are formed.

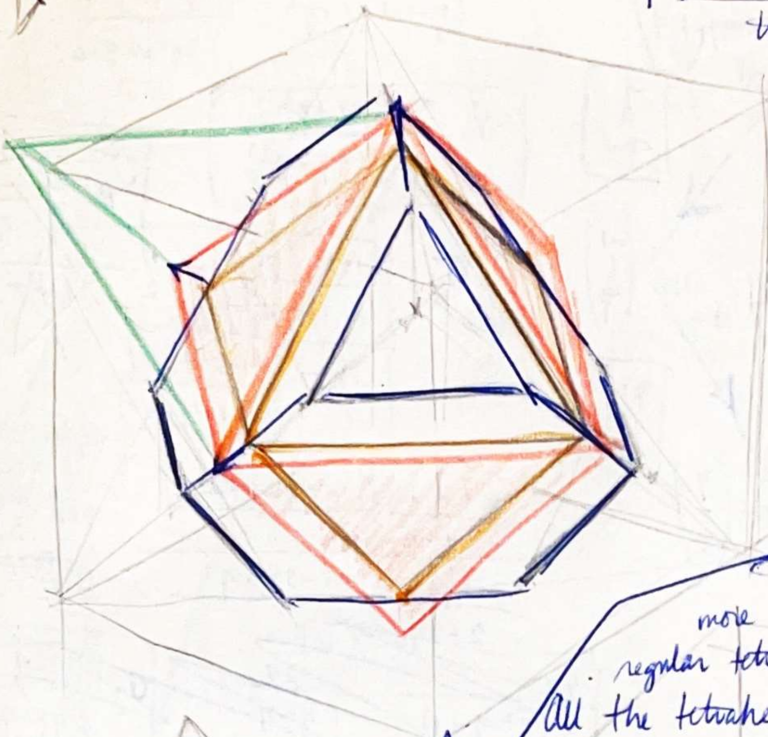
(In the meantime, ~~the~~ faces are not regular, and the hexagons are 12-gons.)



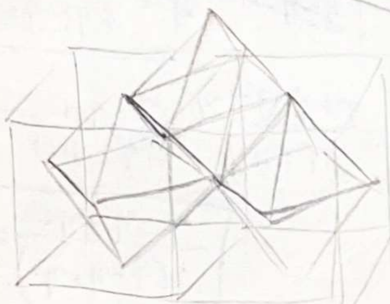




The final state of $\sigma_2[\{6,6\}_D]$ is
an assembly of plane triangles joined at their vertices and edges.
They are actually larger than — and displaced perpendicularly outward from — the orange triangle shown here.
The edges are exactly twice as long as the original ones.
Perhaps the whole assembly is related to one of Apsimon's figures.
I'll have to check his work to see.
It certainly does appear highly likely that this is an assembly of triangles spanning a symmetrical subset of the Δ circuits of the f.c.c. graph,
but this is only a conjecture.
~~(I'm sure it's correct!)~~



No, on second thought, it looks more like a [f.c.c.] assembly of regular tetrahedra joined at their vertices (?).
All the tetrahedra are identically oriented.
☆ They form a set of half of all the tetrahedra in the f.c.c. packing of octahedra and tetrahedra.



The half that's missing is the set of tetrahedra which sit on the open faces of the orange octahedra.

This appears to be a "vertically regular" polyhedron, in the sense that all vertices are equiv., all edges are equiv., and all faces are equiv., and each face is adjoined to only one other at a given edge. However, the "vertex figure" at any vertex is a tetrahedral cluster of four separated equilateral triangles!

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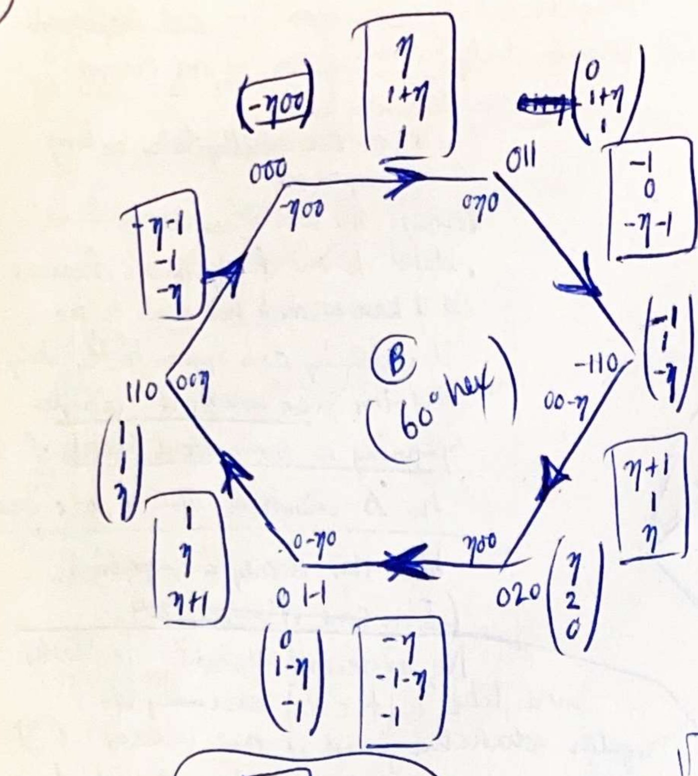


$$\cos\varphi_B = \frac{\begin{pmatrix}\eta\\1+\eta\\1\end{pmatrix}\begin{pmatrix}\eta+1\\1\\\eta\end{pmatrix}}{(\eta+1)^2+1+\eta^2} = \frac{\eta^2+\eta+1+\eta+\eta}{\eta^2+2\eta+1+1+\eta^2} = \frac{\eta^2+3\eta+1}{2\eta^2+2\eta+2}$$

$$\boxed{\cos\varphi_B = \frac{1+3\eta+\eta^2}{2(1+\eta+\eta^2)}}$$

$$\therefore \sigma_B = \sqrt{\frac{\frac{1+3\eta+\eta^2}{2(1+\eta+\eta^2)}+\frac{1}{2}}{1-\frac{(1+3\eta+\eta^2)}{2(1+\eta+\eta^2)}}}$$

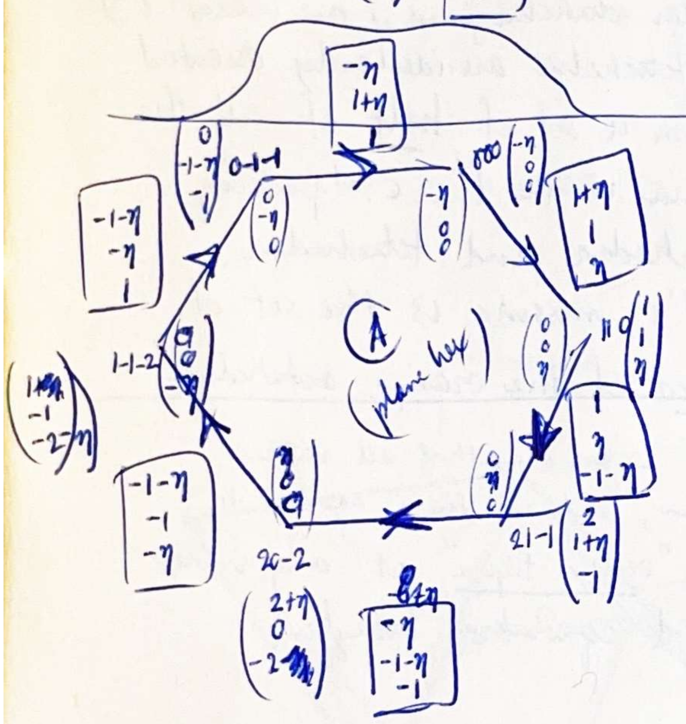
$$= \frac{\frac{2+6\eta+2\eta^2+2+2\eta+2\eta^2}{4+4\eta+4\eta^2}}{\frac{2+2\eta+2\eta^2-1-3\eta-\eta^2}{2+2\eta+2\eta^2}}$$

$$= \cancel{\sqrt{\frac{4\eta}{2(\eta^2-\eta+1)}}} \quad \cancel{\frac{2\eta}{\eta^2-\eta+1}} \quad \cancel{\sigma_B = \sqrt{\frac{2\eta}{\eta^2-\eta+1}}}$$

$$\sqrt{\frac{4\eta^2+8\eta+4}{2(\eta^2-\eta+1)}} = \sqrt{2}\sqrt{\frac{\eta^2+2\eta+1}{\eta^2-\eta+1}}$$

When $\sigma_B = \frac{1}{\sqrt{2}}$, $\cancel{\eta = \ldots}$

$$\boxed{\sigma_B = \frac{\sqrt{2}(1+\eta)}{\sqrt{1-\eta+\eta^2}}}$$



$$\cos\varphi_A = \frac{\begin{pmatrix}1+\eta\\1\\\eta\end{pmatrix}\begin{pmatrix}\eta\\-1-\eta\\-1\end{pmatrix}}{2\eta^2+2\eta+2} = \frac{\eta^2+\eta-\eta-1-\eta}{2(\eta^2+\eta+1)} = \frac{\eta^2-\eta-1}{2(\eta^2+\eta+1)} = \cos\varphi_A$$

$$\boxed{\frac{-1-\eta+\eta^2}{2(1+\eta+\eta^2)} = \cos\varphi_A}$$

$$\therefore \sigma_A = \sqrt{\frac{\frac{-1-\eta+\eta^2}{2(1+\eta+\eta^2)}+\frac{1}{2}}{1+\frac{1+\eta-\eta^2}{2(1+\eta+\eta^2)}}}$$

$$= \sqrt{\frac{-1-\eta+\eta^2+1+\eta+\eta^2}{2+2\eta+2\eta^2+1+\eta-\eta^2}} \quad \frac{2\eta^2}{\eta^2+3\eta+3}$$

$$\text{or } \sigma_A = \sqrt{\frac{2\eta^2}{\eta^2+3\eta+3}}$$

$$\therefore \cancel{\sigma_A = \sqrt{3\eta}}$$

$$\therefore \boxed{\begin{aligned}\sigma_A &= \eta\sqrt{\frac{2}{\eta^2+3\eta+3}} = \frac{\sqrt{2}\,\eta}{\sqrt{\eta^2+3\eta+3}} \\ \sigma_B &= \frac{\sqrt{2}(1+\eta)}{\sqrt{1-\eta+\eta^2}} = \frac{\sqrt{2}(1+\eta)}{\sqrt{1-\eta+\eta^2}}\end{aligned}}$$

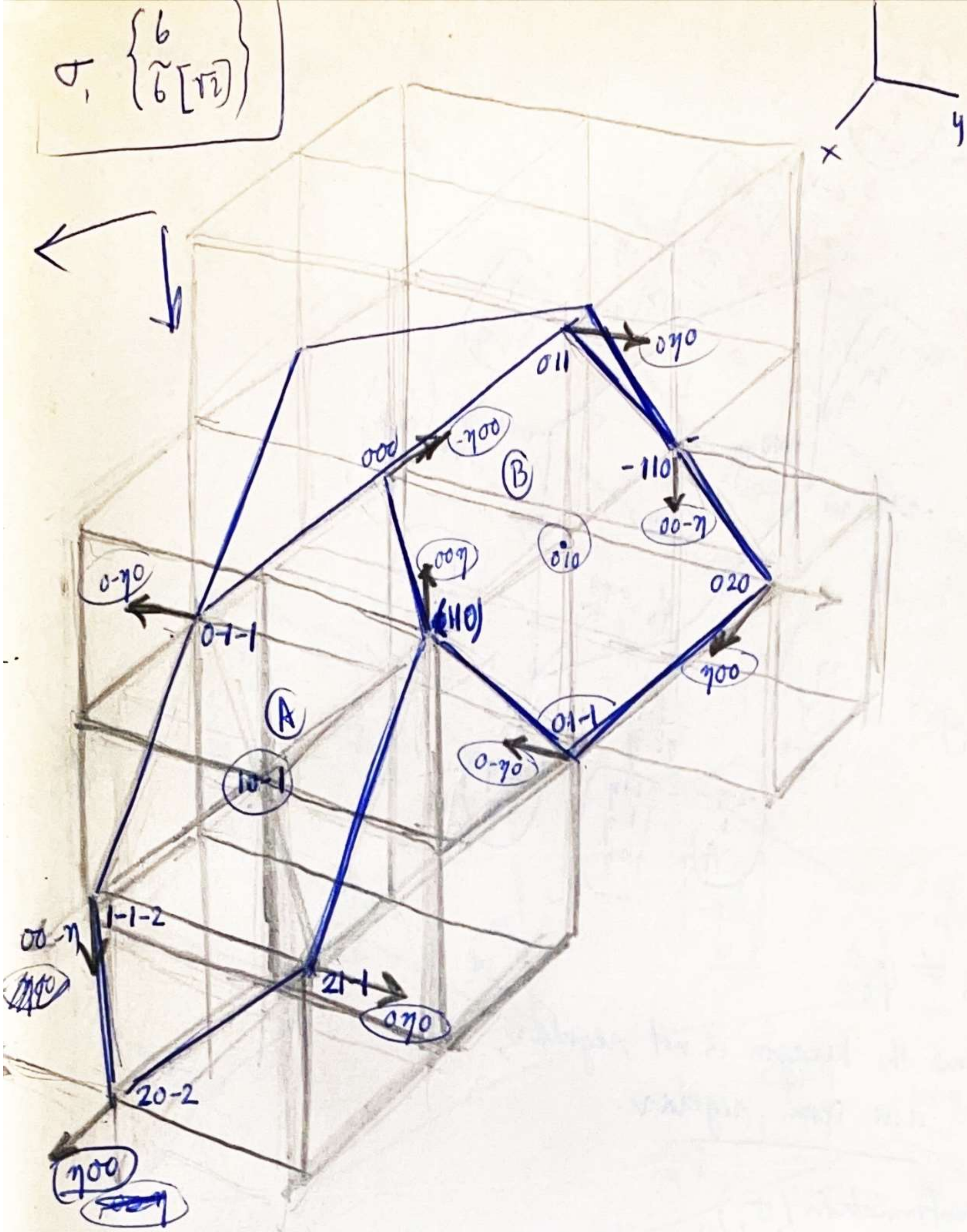
no real roots

<u>Note</u>: If $\sigma_A = \sigma_B$,
then $K(\eta) = 2\eta^3+3\eta^2+3\eta+1 = 0$
This has the solution $\eta = -\frac{1}{2}$, for which

$|\sigma_A| = |\sigma_B| = \sqrt{\frac{2}{7}}$ Then $\boxed{\varphi_A = \varphi_B = 99.953°}$ But this is $\{\tilde{6},\tilde{4}\}_L$, of course.

There are no other real solutions of $\sigma_A = \sigma_B$ (only $\eta = \boxed{-\frac{1}{2} \pm \frac{\sqrt{-3}}{2}}$).

$\sigma_1 \begin{Bmatrix} b \\ \tilde{b}\,[\sqrt{2}] \end{Bmatrix}$



$$\cos\varphi_6 = \frac{\begin{pmatrix}1-\eta\\-2\eta\\1+\eta\end{pmatrix}\begin{pmatrix}-2\eta\\1-\eta\\-(1+\eta)\end{pmatrix}}{(1-\eta)^2+(1+\eta)^2+4\eta^2}$$

$$= -2\eta+2\eta^2-2\eta+2\eta^2-[1+2\eta+\eta^2]$$

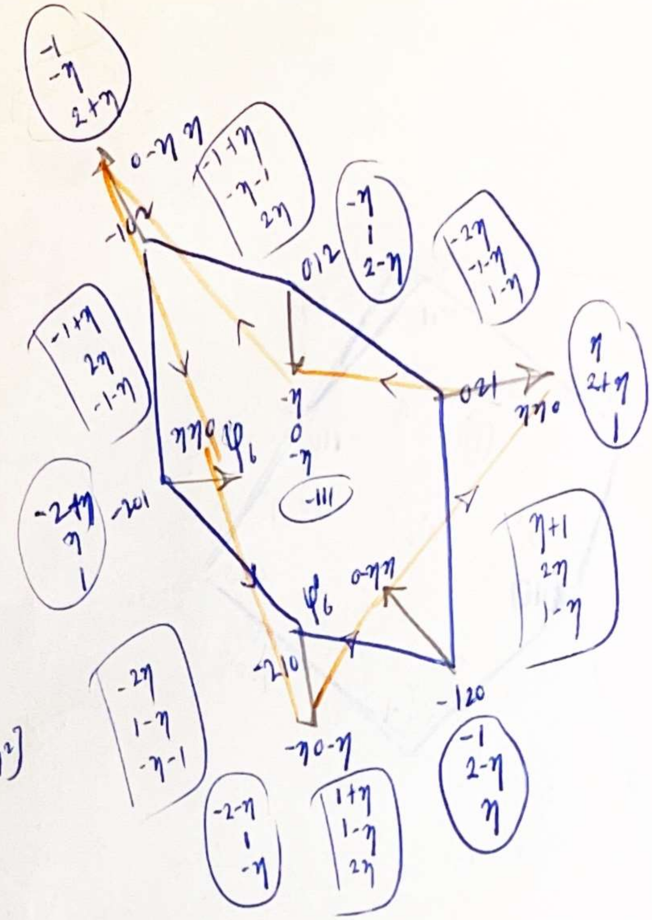
$$= \frac{-6\eta+3\eta^2-1}{1+3\eta^2}$$

$$\boxed{\cos\varphi_6 = \frac{-1-6\eta+3\eta^2}{1+3\eta^2}}$$

$$\cos\varphi_6' = \frac{\begin{pmatrix}2\eta\\-(1-\eta)\\1+\eta\end{pmatrix}\begin{pmatrix}1+\eta\\1-\eta\\2\eta\end{pmatrix}}{1+3\eta^2} \qquad \begin{matrix}2\eta+2\eta^2+2\eta+2\eta^2\\ -\left[(1-\eta)^2 = 1-2\eta+\eta^2\right]\end{matrix}$$

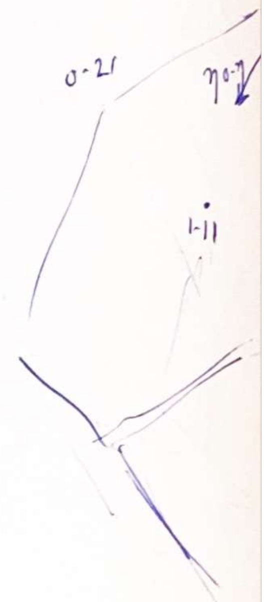
$$= \frac{-1+6\eta+3\eta^2}{1+3\eta^2}$$

$$\boxed{\cos\varphi_6' = \frac{-1+6\eta+3\eta^2}{1+3\eta^2}}$$



$\varphi_6 \neq \varphi_6'$

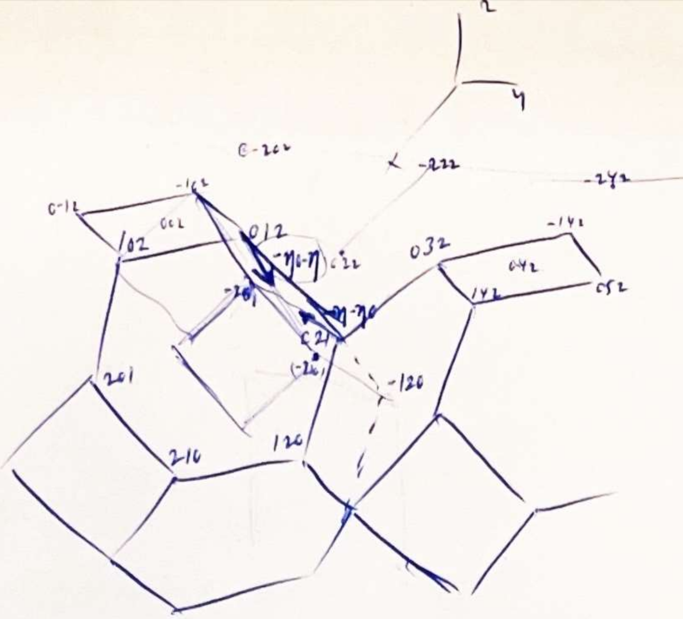
Hence the hexagon is not regular, it is semi-regular.

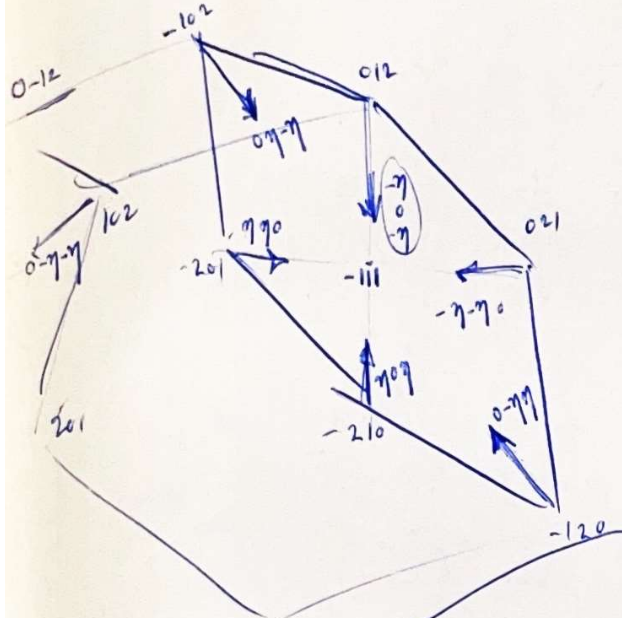


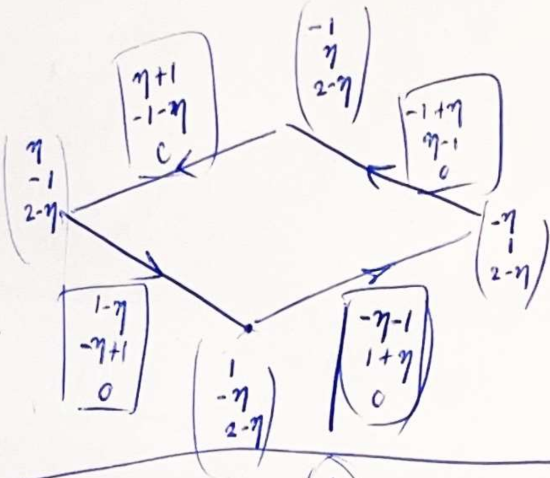
Thus — in this transformation (σ_1), the quadrilaterals remain regular, but the hexagons do not.

Semi-regular figure

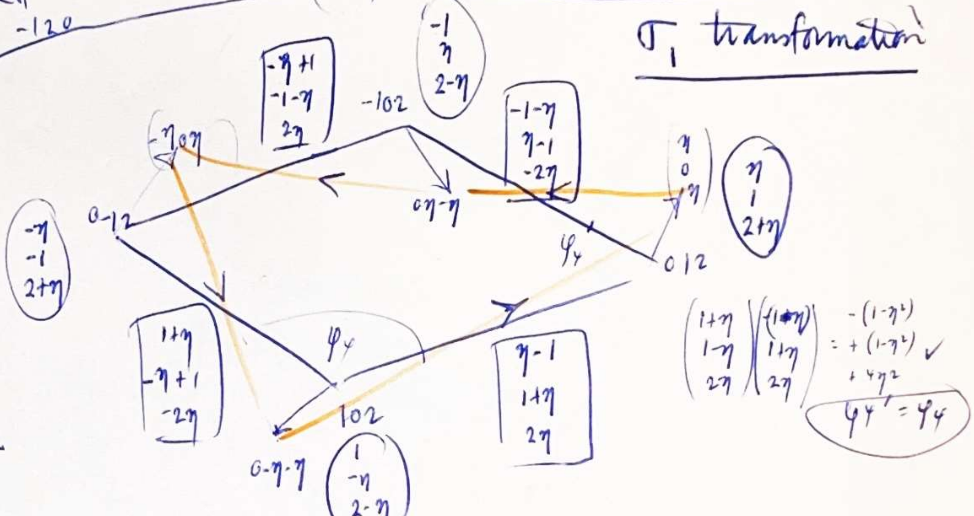
from $\begin{Bmatrix}4\\6\end{Bmatrix}$ by σ_2







σ_1 transformation



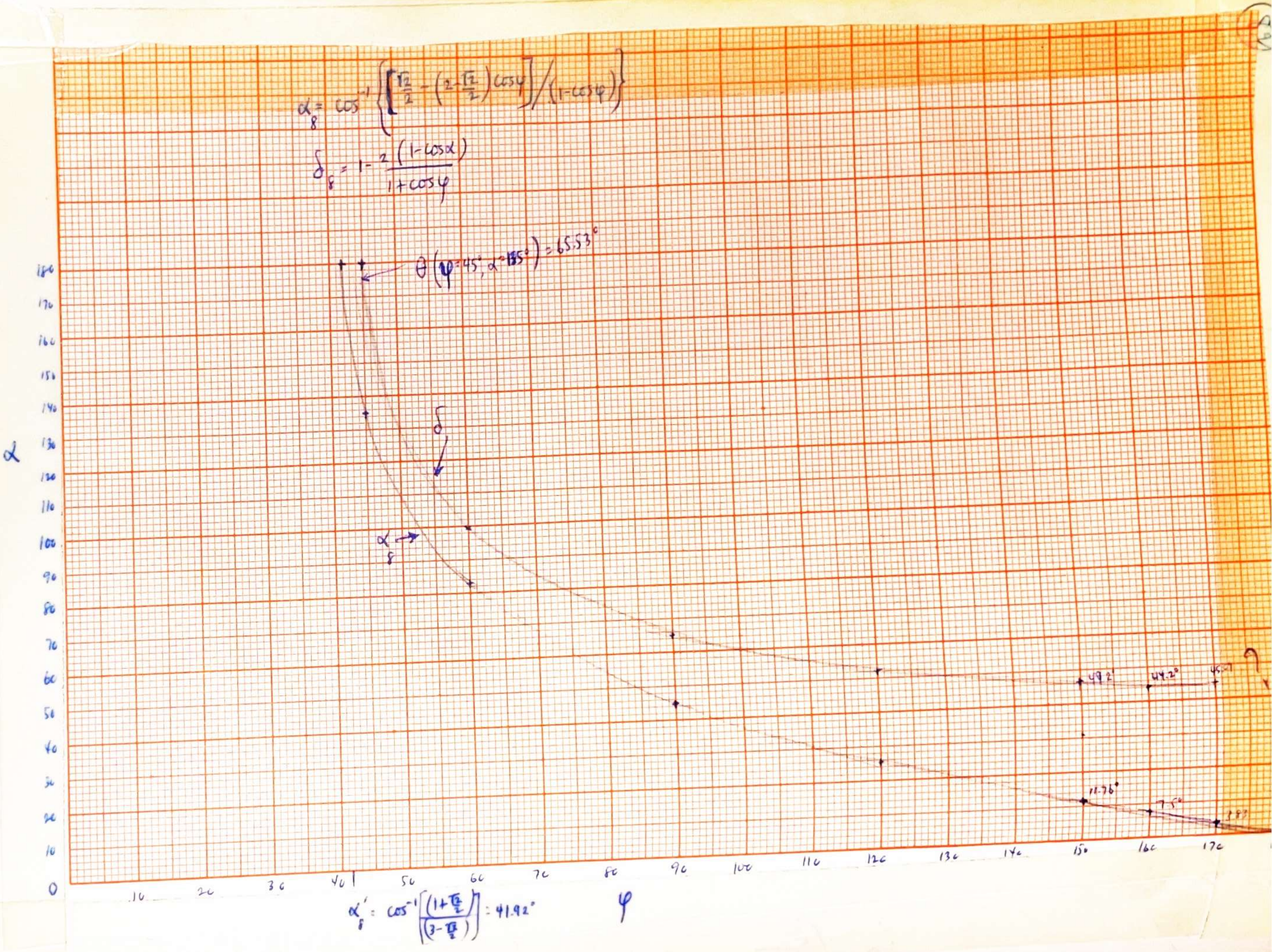
$$\cos\varphi_4 = \frac{\begin{pmatrix}\eta-1\\1+\eta\\2\eta\end{pmatrix}\begin{pmatrix}-(\eta+1)\\-(1+\eta)\\2\eta\end{pmatrix}}{(1+\eta)^2+(1-\eta)^2+4\eta^2}$$

$$= \frac{1-\eta^2-[1-\eta^2]+4\eta^2}{1+2\eta+\eta^2+1-2\eta+\eta^2+4\eta^2} = \frac{4\eta^2}{2+6\eta^2}$$

$$\therefore \boxed{\cos\varphi_4 = \frac{2\eta^2}{1+3\eta^2}}$$

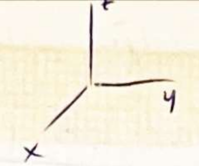
$$\begin{pmatrix}1+\eta\\1-\eta\\2\eta\end{pmatrix}\begin{pmatrix}-(1+\eta)\\1+\eta\\2\eta\end{pmatrix} \begin{matrix}-(1-\eta^2)\\=+(1-\eta^2)\ \checkmark\\+4\eta^2\end{matrix}$$

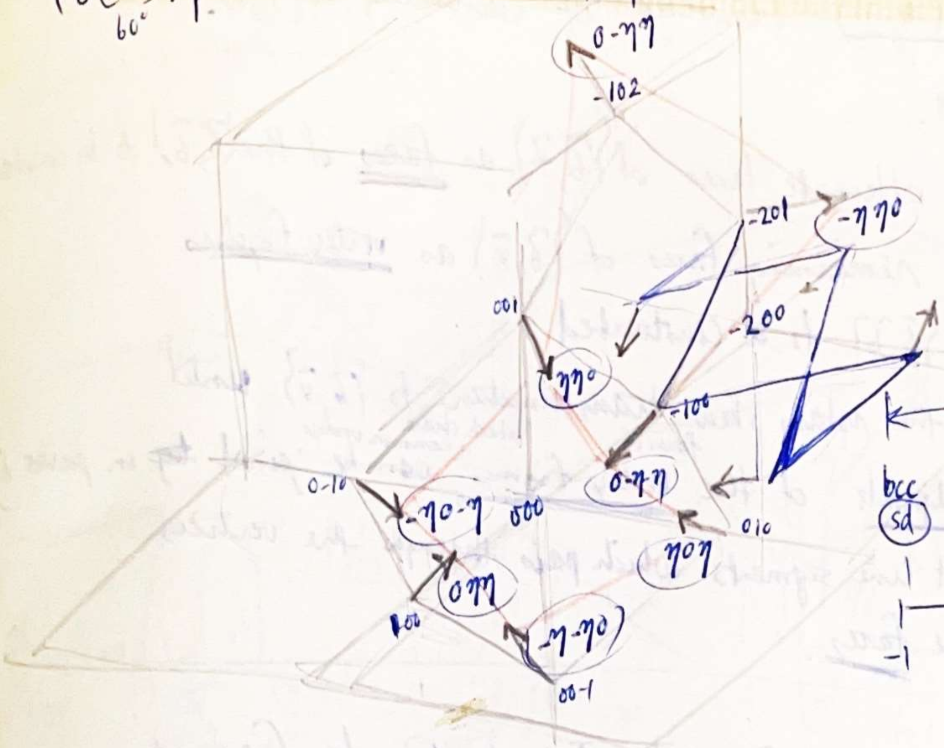
$\varphi_4' = \varphi_4$

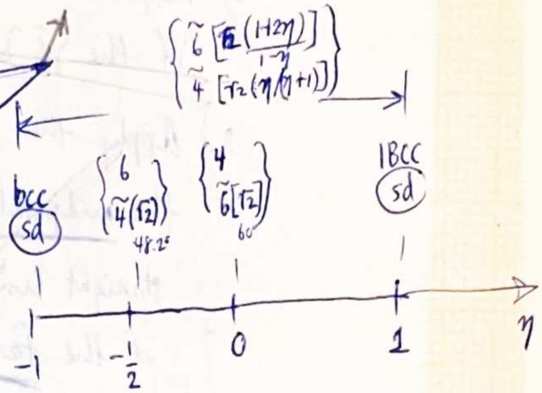


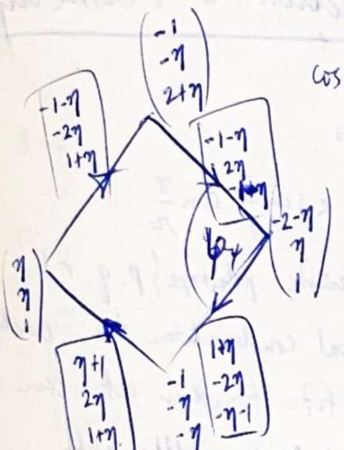
$\alpha_8 = \cos^{-1}\left\{\left[\frac{\sqrt{2}}{2} - \left(2-\frac{\sqrt{2}}{2}\right)\cos\varphi\right] \Big/ (1-\cos\varphi)\right\}$
$\delta_8 = 1 - 2\frac{(1-\cos\alpha)}{1+\cos\varphi}$
$\theta(\varphi=45°, \alpha=135°) = 65.53°$
δ
α_8
α
φ
49.2°
44.2°
11.76°
$\alpha'_8 = \cos^{-1}\left[\frac{(1+\frac{\sqrt{2}}{2})}{(3-\frac{\sqrt{2}}{2})}\right] = 41.92°$

$$\sigma_1 \left\{ \begin{matrix} 4 \\ \tilde{6}[\sqrt{2}] \\ 60^\circ \end{matrix} \right\}_p \longrightarrow \left\{ \begin{matrix} 4[\sqrt{2}\eta/(1+\eta)] \\ \tilde{6}[\sqrt{2}(1+2\eta)/(1-\eta)] \end{matrix} \right\}_p$$





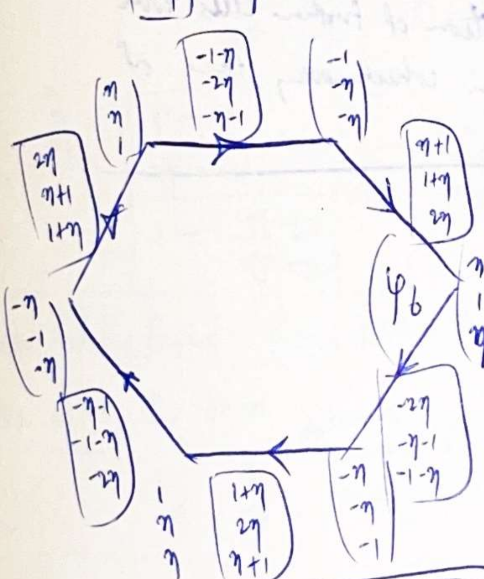




$$\cos\varphi_4 = \frac{\begin{pmatrix}1+\eta \\ -2\eta \\ -1-\eta\end{pmatrix}\begin{pmatrix}1+\eta \\ -2\eta \\ 1+\eta\end{pmatrix}}{2(1+\eta)^2+4\eta^2} = \frac{1+2\eta+\eta^2+4\eta^2-[1+2\eta+\eta^2]}{2+4\eta+2\eta^2+4\eta^2} = \frac{4\eta^2}{2+4\eta+6\eta^2} = \boxed{\frac{2\eta^2}{1+2\eta+3\eta^2}}$$

$(5\eta+1)(\eta+1)$

$$\cos\varphi_6 = \frac{\begin{pmatrix}1+\eta \\ 2\eta \\ 1+\eta\end{pmatrix}\begin{pmatrix}1+\eta \\ 1+\eta \\ 2\eta\end{pmatrix}}{2+4\eta+6\eta^2} = \frac{1+2\eta+\eta^2+2\eta+2\eta^2+2\eta+2\eta^2}{2+4\eta+6\eta^2} = \frac{1+6\eta+5\eta^2}{2+4\eta+6\eta^2} = \frac{(1+\eta)(1+5\eta)}{2[1+2\eta+3\eta^2]}$$



$$\sigma_4 = \sqrt{\frac{\frac{2\eta^2}{1+2\eta+3\eta^2} - 0}{1-\frac{2\eta^2}{1+2\eta+3\eta^2}}} = \sqrt{\frac{2\eta^2}{1+2\eta+3\eta^2-2\eta^2}} = \sqrt{\frac{2\eta^2}{1+2\eta+\eta^2}}$$

$$\boxed{\sigma_4 = \sqrt{2}\left(\frac{\eta}{\eta+1}\right)}$$

$$\sigma_6 = \sqrt{\frac{\frac{1+6\eta+5\eta^2}{2+4\eta+6\eta^2}+\frac{1}{2}}{1-\frac{1+6\eta+5\eta^2}{2+4\eta+6\eta^2}}} = \sqrt{\frac{1+6\eta+5\eta^2+1+2\eta+3\eta^2}{2+4\eta+6\eta^2-1-6\eta-5\eta^2}}$$

$2+8\eta+8\eta^2$

$1+4\eta+4\eta^2$

$(1+2\eta)$

$1-2\eta+\eta^2$

$$\boxed{\sigma_6 = \frac{\sqrt{2}(1+2\eta)}{1-\eta}}$$

When $\eta = -\frac{1}{2}$, $\sigma_6 = 0$, $\sigma_4 = \sqrt{2}\ (48.2^\circ)$

This is $t\{6, \tilde{4}\}_p$

When $\eta = 1$, $\sigma_6 \to \infty$, $\sigma_4 \to \frac{1}{\sqrt{2}}\ (\varphi_4 = 70.5^\circ)$

This is space-filling of tetragonal tetrahedron sym. domains surrounding the [IBCC-distributed] 4-faced cubic holes in $\{4, \tilde{6}\}$

When $\eta = -1$, $\sigma_4 \to \infty$, $\sigma_6 = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$ $(\varphi_4 = 90^\circ)$

This is space filling of expanded octahedron sym. domains surrounding the [b.c.c.-distributed] faceless cubic holes in $\{4, \tilde{6}\}$

(84) An alternative method of deriving $\{\tilde{6}, \tilde{6}\}$

is to start with $\{\tilde{6}, \tilde{4}\}$. (instead of constructing $h\{\tilde{4}, \tilde{6}\}$.

The procedure is:

1) Choose alternate faces of $\{\tilde{6}, \tilde{4}\}$ as faces of the $\{\tilde{6}, \tilde{6}\}$ to be constructed.
2) Regard remaining faces of $\{\tilde{6}, \tilde{4}\}$ as vertex figures of the $\{\tilde{6}, \tilde{6}\}$ to be constructed
3) Apply the rotary skew transformation to $\{\tilde{6}, \tilde{4}\}$ until the midpoints of the faces of vertex figures which share a common vertex can be joined ~~to~~ in pairs by straight line segments which pass through the vertices of the faces.

See p. 401 in Coxeter's Introduction to Geometry:

§22.3: Construction for Regular Polytopes.

~~The~~ "We have seen that the inequality 22.21 : $\cos\frac{\pi}{q} < \sin\frac{\pi}{p}\sin\frac{\pi}{r}$

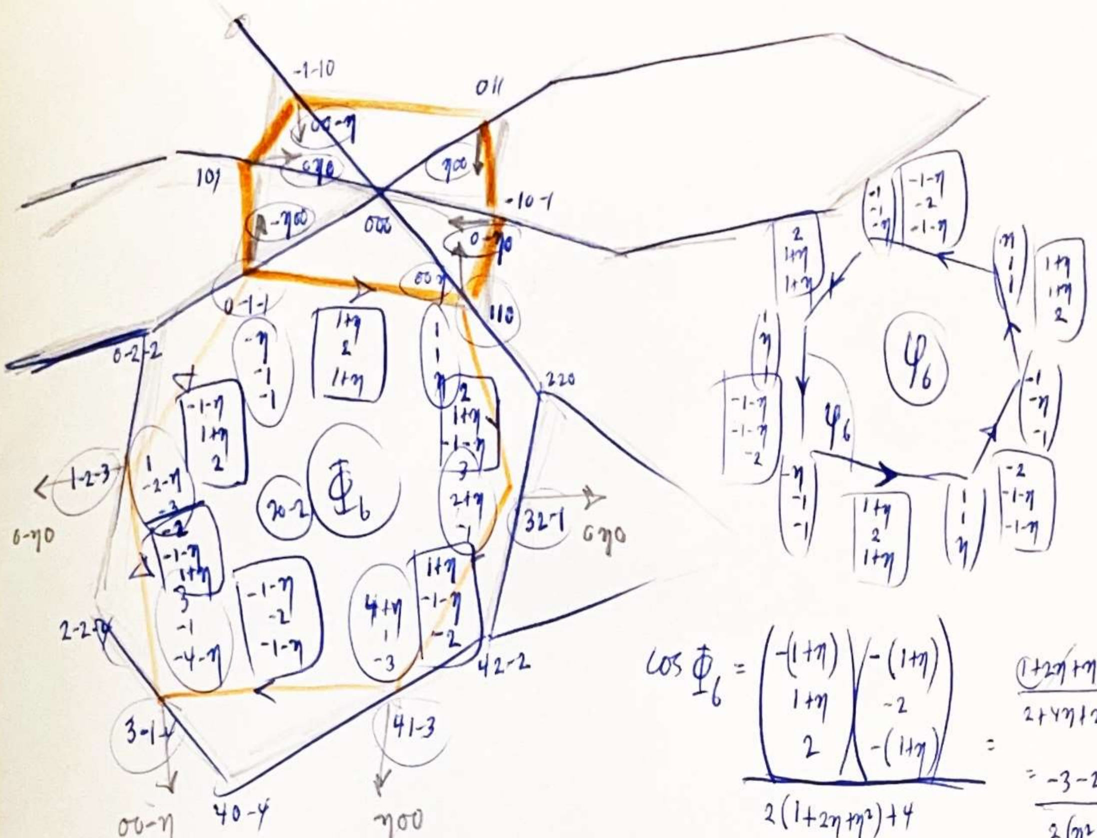
is a necessary condition for the existence of a finite polytope $\{p, q, r\}$.

The sufficiency of the condition requires an actual construction for each of the 6 figures. We know that r cells can fit together at an edge, but it is not obvious that the addition of further cells will ultimately yield a closed configuration, in which every face of every cell belongs also to another cell."

2

σ transf. on $\left\{ \begin{matrix} 6 \\ \tilde{6}\,[\sqrt{8}]_{33.6} \end{matrix} \right\}_P \longrightarrow \left\{ \begin{matrix} 6\left[\sqrt{2}(2+\eta)/(1-\eta)\right] \\ \tilde{6}\left[\sqrt{2}(\eta/3+\eta)\right] \end{matrix} \right\} \longrightarrow 4$

$$\cos\varphi_6 = \frac{\begin{pmatrix}1+\eta\\1+\eta\\2\end{pmatrix}\begin{pmatrix}1+\eta\\2\\1+\eta\end{pmatrix}}{2(1+\eta)^2+4} = \frac{1+2\eta+\eta^2+2+2\eta+2+2\eta}{2(1+2\eta+\eta^2)+4} = \frac{5+6\eta+\eta^2}{6+4\eta+2\eta^2} = \frac{(\eta+5)(\eta+1)}{2(\eta^2+2\eta+3)}$$



$$\cos\Phi_6 = \frac{\begin{pmatrix}-(1+\eta)\\1+\eta\\2\end{pmatrix}\begin{pmatrix}-(1+\eta)\\-2\\-(1+\eta)\end{pmatrix}}{2(1+2\eta+\eta^2)+4} = \frac{1+2\eta+\eta^2-2-2\eta-2-2\eta}{2+4\eta+2\eta^2+4} = \frac{-3-2\eta+\eta^2}{2(\eta^2+2\eta+3)} = \frac{(\eta-3)(\eta+1)}{2(\eta^2+2\eta+3)}$$

$$\therefore\ \sigma_{\varphi_6} = \sqrt{\frac{\frac{(\eta+5)(\eta+1)}{2(\eta^2+2\eta+3)}+\frac{1}{2}}{1-\frac{5+6\eta+\eta^2}{2(\eta^2+2\eta+3)}}} = \sqrt{\frac{\eta^2+6\eta+5+\eta^2+2\eta+3}{2\eta^2+4\eta+6-5-6\eta-\eta^2}} = \sqrt{\frac{2\eta^2+8\eta+8}{\eta^2-2\eta+1}} = \sqrt{\frac{2(\eta^2+4\eta+4)}{(\eta-1)^2}} = \boxed{\sqrt{2}\,\frac{(2+\eta)}{(1-\eta)}}\ ✓$$

$$\sigma_{\Phi_6} = \sqrt{\frac{\frac{\eta^2-2\eta-3}{2(\eta^2+2\eta+3)}+\frac{1}{2}}{1-\frac{\eta^2-2\eta-3}{2(\eta^2+2\eta+3)}}} = \frac{\eta^2\cancel{-2\eta}\cancel{-3}+\eta^2\cancel{+2\eta}\cancel{+3}}{2\eta^2+4\eta+6-\eta^2+2\eta+3} = \frac{2\eta^2}{\eta^2+6\eta+9} = \frac{2\eta^2}{(\eta+3)^2} = \boxed{\frac{\sqrt{2}\,\eta}{3+\eta}}\ ✓$$

BUT SEE PAGE 43!

We have finally found a proper skewing transformation that leads to a space filling of ◊ tetrahedra!

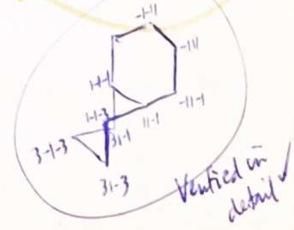
When $\eta=1$, $\sigma_{\varphi_6}\to\infty$, $\sigma_{\Phi_6}=\frac{1}{\sqrt{8}}$, i.e., $\Phi_6 \simeq 109.5°$ — diamond tetrahedra space filling

When $\eta=-2$, $\sigma_{\varphi_6}=0$, $\sigma_{\Phi_6}=-2\sqrt{2}=-\sqrt{8}$, $\Phi_6 = 33.6°$

Thus, $\eta=-2$ just interchanges the plane and skew hexagons!

When $\eta=-1$: $|\sigma_{\varphi_6}| = |\sigma_{\Phi_6}| = \frac{\sqrt{2}}{2}$ $\therefore$ $\varphi_6 = \varphi_6' = 90°$

Then this becomes the ◊ IPMS $\{\tilde{6},4\}$!! ($\eta=-1$)



Verified in detail ✓

σ_1 on $\left\{\begin{matrix}6\\ \tilde{4}[\quad]\end{matrix}\right\}_p \to \left\{\begin{matrix}\tilde{4}\left[\sqrt{2}(1+\eta)/(1-\eta)\right]\\ \tilde{6}\left[2\sqrt{2}\,\eta/(\eta+3)\right]\end{matrix}\right\}_p$

(referred to coordinates expressed in smallest integers)

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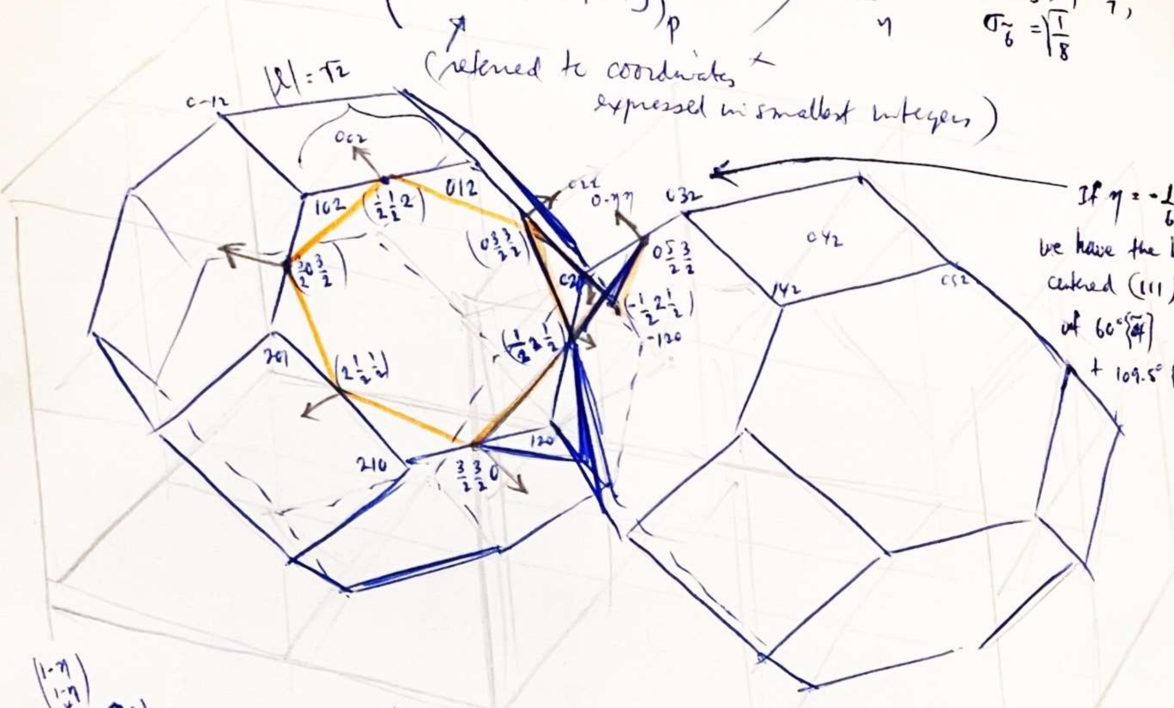
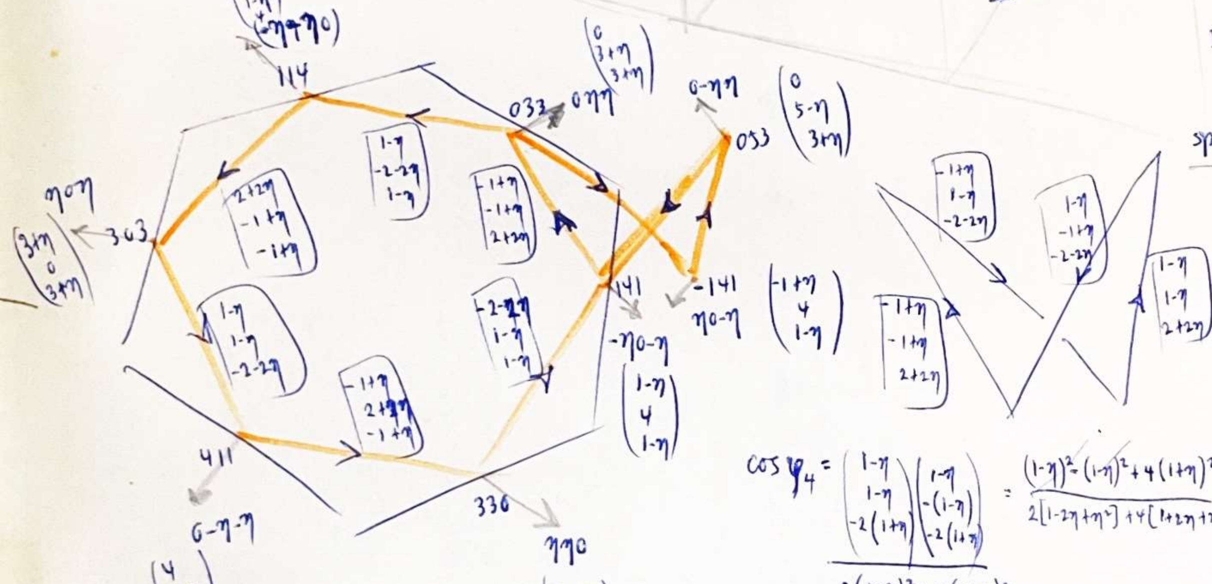
For $\eta=-\frac{1}{3}$, $\eta=\frac{3}{7}$, $\sigma_{\tilde{6}}=\sqrt{\frac{1}{8}}$



If $\eta=-\frac{1}{6}$, we have the locally centered (111) of $60°\{\tilde{4}\}$ + $109.5°\{6\}$



If $\eta=1$, $\sigma_4\to\infty$

$\sigma_6=\frac{1}{\sqrt{2}}$

space-filling of exp. octahedra

If $\eta=-3$, $\sigma_6\to\infty$

$\sigma_4=\frac{1}{\sqrt{2}}$

space-filling of tetrag. tetrahedra

$$\cos\psi_4=\frac{\begin{pmatrix}1-\eta\\1-\eta\\-2(1+\eta)\end{pmatrix}\begin{pmatrix}1-\eta\\-(1-\eta)\\-2(1+\eta)\end{pmatrix}}{2(1-\eta)^2+4(1+\eta)^2}=\frac{(1-\eta)^2-(1-\eta)^2+4(1+\eta)^2}{2[1-2\eta+\eta^2]+4[1+2\eta+\eta^2]}=\frac{4(1+\eta)^2}{2-4\eta+2\eta^2+4+8\eta+4\eta^2}=\frac{4(1+\eta)^2}{6+4\eta+6\eta^2}=\frac{2(1+\eta)^2}{3+2\eta+3\eta^2}$$

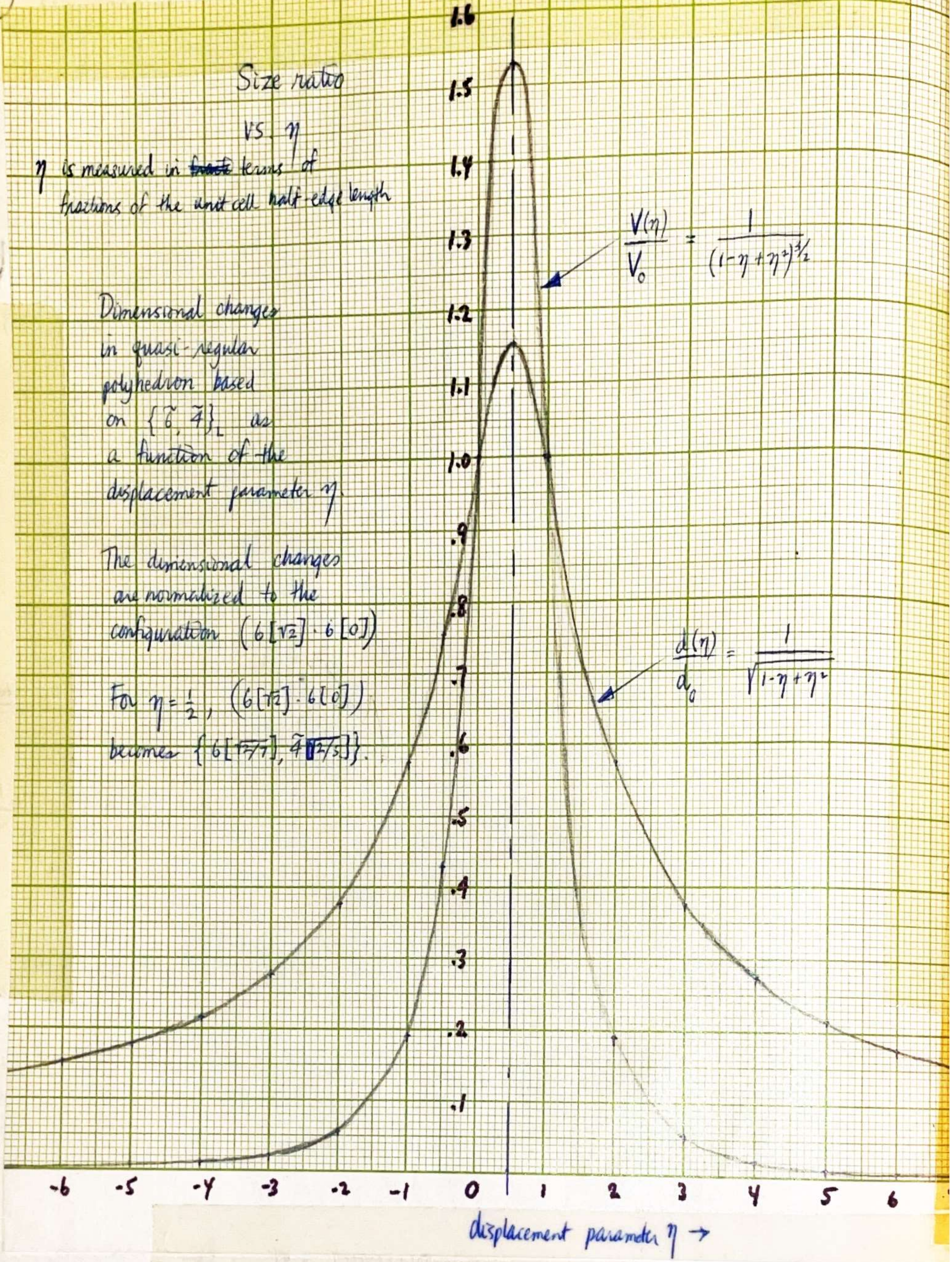
$$\sigma_4=\sqrt{\frac{\frac{2+4\eta+2\eta^2}{3+2\eta+3\eta^2}}{1-\frac{2+4\eta+2\eta^2}{3+2\eta+3\eta^2}}}=\sqrt{\frac{2+4\eta+2\eta^2}{3+2\eta+3\eta^2-2-4\eta-2\eta^2}}=\sqrt{\frac{2(1+\eta)^2}{1-2\eta+\eta^2}}=\sqrt{\frac{2(1+\eta)^2}{(1-\eta)^2}}=\boxed{\sqrt{2}\left(\frac{1+\eta}{1-\eta}\right)}$$

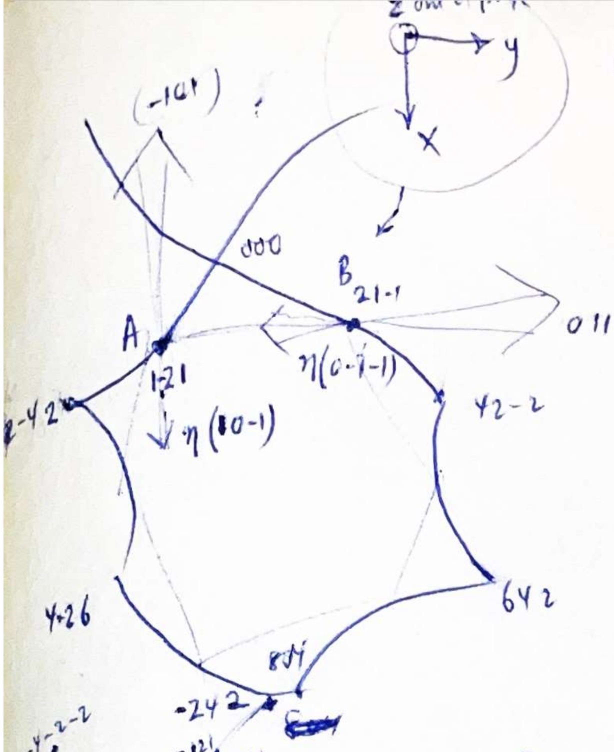
σ_4

$$\cos\psi_6=\frac{\begin{pmatrix}-(1-\eta)\\-(1-\eta)\\2(1+\eta)\end{pmatrix}\begin{pmatrix}-(1-\eta)\\2(1+\eta)\\-(1-\eta)\end{pmatrix}}{6+4\eta+6\eta^2}=\frac{(1-\eta)^2-2(1-\eta^2)-2(1-\eta^2)}{6+4\eta+6\eta^2}=\frac{1-2\eta+\eta^2-2+2\eta^2-2+2\eta^2}{6+4\eta+6\eta^2}=\frac{-3-2\eta+5\eta^2}{6+4\eta+6\eta^2}$$

$$\sigma_6=\sqrt{\frac{\frac{-3-2\eta+5\eta^2}{6+4\eta+6\eta^2}+\frac{1}{2}}{1-\frac{(-3-2\eta+5\eta^2)}{6+4\eta+6\eta^2}}}=\frac{-3-2\eta+5\eta^2+3+2\eta+3\eta^2}{6+4\eta+6\eta^2+3+2\eta-5\eta^2}=\frac{8\eta^2}{9+6\eta+\eta^2}$$

$$=\sqrt{\frac{8\eta^2}{(\eta+3)^2}}=\boxed{\frac{2\sqrt{2}\,\eta}{\eta+3}}=\sigma_6$$


Size ratio
vs. η
η is measured in terms of fractions of the unit cell half-edge length
Dimensional changes in quasi-regular polyhedron based on {6̃, 4̃}_L as a function of the displacement parameter η.
The dimensional changes are normalized to the configuration (6[√2]·6[0])
For η = 1/2, (6[√2]·6[0]) becomes {6[√7/7], 4̃[√2/3]}.
V(η)/V_0 = 1/(1-η+η²)^{3/2}
d(η)/d_0 = 1/√(1-η+η²)
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6
displacement parameter η →



Let $\eta = 1/3$ ~~(see coordinates, on p. 89)~~

Then $(1\ -2\ 1)$ $\quad$ $(2\ 1\ -1)$

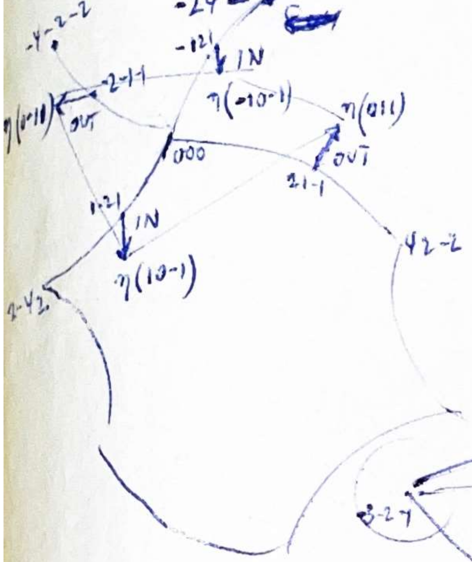
$+\ (1/3\ 0\ -1/3)$ $\quad$ $+\ (0\ +1/3\ +1/3)$

$= 4/3\ -2\ 2/3$ $\quad$ $= 2\ 4/3\ -2/3$

$= \frac{1}{3}(4\ -6\ 2)$ $\quad$ $= \frac{1}{3}(6\ 4\ -2)$

Diff. $= \frac{1}{3}(2\ 10\ -4)$

$= \frac{2}{3}(1\ 5\ -2)$

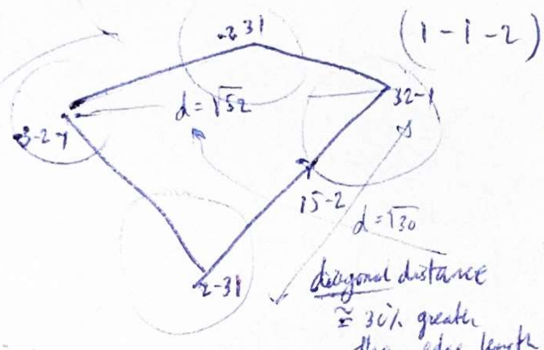


Then $\vec{\eta}_A = (1/3\ 0\ -1/3) = \frac{1}{3}(1\ 0\ -1)$

$\vec{\eta}_B = (0\ 1/3\ 1/3) = \frac{1}{3}(0\ 1\ 1)$

$\vec{\eta}_A - \vec{\eta}_B = \frac{1}{3}(1\ -1\ -2)\ (\propto \vec{\Lambda}_{BA})$

$(1\ -1\ -2)\cdot(1\ 5\ -2) = 1-5+4 = 0$



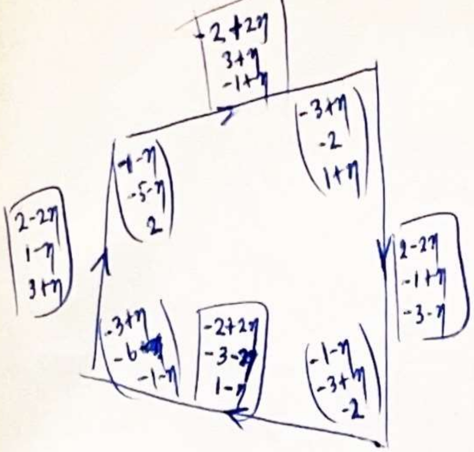
i.e., for $\eta = 1/3$ (locally centered graph)

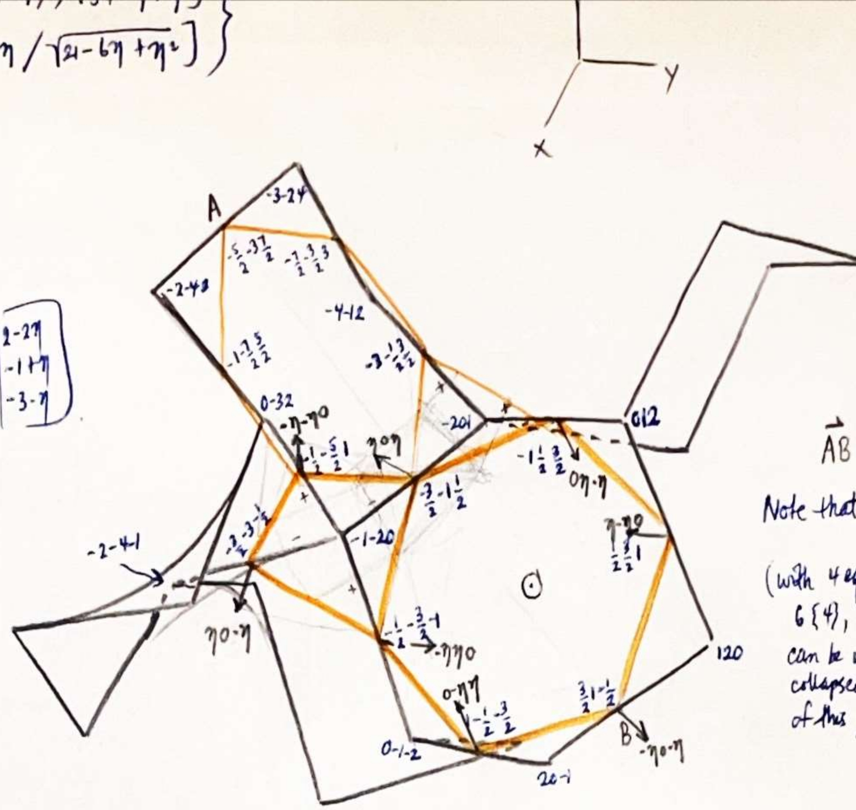
$\vec{\Lambda}_{ij} \cdot \vec{r}_{ij} = 0$

The question is:

Why does it happen that $\vec{\Lambda}_{ij} \cdot \vec{r}_{ij} = 0$ only for the locally-centered configuration of the quasi-regular graph?

$$\sigma_1 \left\{ \begin{matrix} 4\,[\sqrt{2/3}]_{73.4°} \\ 6 \end{matrix} \right\}_L = \left\{ \begin{matrix} [illegible] \\ \tilde{6}\,[2\sqrt{2}\,\eta / \sqrt{21-6\eta+\eta^2}] \end{matrix} \right\}$$

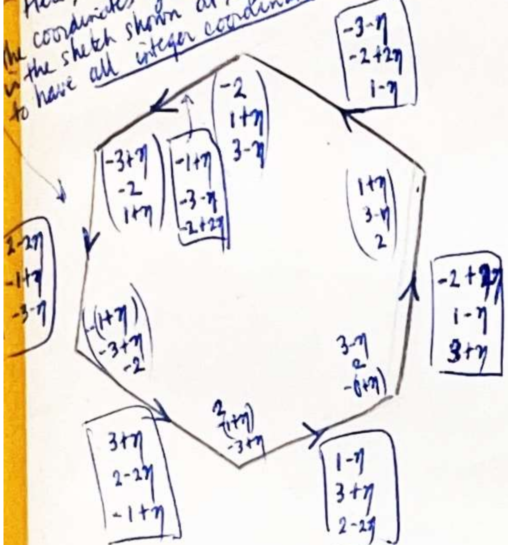




$\vec{AB} = (4\ 4\ {-4})$

Note that the $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ (with 4 equatorial $\{6\}$ and $6\{4\}$, based on $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$) can be inscribed in the collapsed state ($\eta \to \infty$) of this graph. ★

Here, double the values of the coordinates given for the vertices in the sketch shown at right in order to have all integer coordinates.



$$\cos\varphi_6 = \frac{\begin{pmatrix} -(3+\eta) \\ -(2-2\eta) \\ (1-\eta) \end{pmatrix}\begin{pmatrix} 2-2\eta \\ -(1-\eta) \\ -(3+\eta) \end{pmatrix}}{(1-\eta)^2+(2-2\eta)^2+(3+\eta)^2} = \frac{\begin{matrix} -6+4\eta+2\eta^2 \\ 2-4\eta+2\eta^2 \\ -3+2\eta+\eta^2 \end{matrix}}{\begin{matrix} 1-2\eta+\eta^2+(4-8\eta+4\eta^2+3) \\ 4-8\eta+4\eta^2 \\ 9+6\eta+\eta^2 \\ 14-4\eta+6\eta^2 = 2(7-2\eta+3\eta^2) \end{matrix}} = \frac{-7+2\eta+5\eta^2}{2(7-2\eta+3\eta^2)}$$

$$\sigma_6 = \left[\frac{\frac{-7+2\eta+5\eta^2}{2(7-2\eta+3\eta^2)} + \frac{1}{2}}{1 - \frac{-7+2\eta+5\eta^2}{2(7-2\eta+3\eta^2)}}\right]^{1/2} = \left[\frac{-7+2\eta+5\eta^2+7-2\eta+3\eta^2}{14-4\eta+6\eta^2+7-2\eta-5\eta^2}\right]^{1/2} = \left[\frac{8\eta^2}{21-6\eta+\eta^2}\right]^{1/2}$$

$$\therefore \boxed{\sigma_6 = \frac{2\sqrt{2}\,\eta}{\sqrt{21-6\eta+\eta^2}}}$$

$\boxed{\text{Cf. } \cos(73.39°) = .2857 = 2/7}$ ✓

$$\cos\varphi_4 = \frac{\begin{pmatrix} -2+2\eta \\ 1-\eta \\ 3+\eta \end{pmatrix}\begin{pmatrix} -2+2\eta \\ -3-\eta \\ 1-\eta \end{pmatrix}}{2(7-2\eta+3\eta^2)} = \frac{\begin{matrix} 4-8\eta+4\eta^2 \\ -3+2\eta+\eta^2 \\ 3-2\eta-\eta^2 \end{matrix}}{2(\quad)} = \frac{4-8\eta+4\eta^2}{2(\quad)} = \frac{2(\eta^2-2\eta+1)}{(\quad)} = \frac{2(1-\eta)^2}{(7-2\eta+3\eta^2)}$$

$$\sigma_4 = \left[\frac{\frac{2-4\eta+2\eta^2}{7-2\eta+3\eta^2}}{1 - \frac{2-4\eta+2\eta^2}{7-2\eta+3\eta^2}}\right]^{1/2} = \left[\frac{2(1-\eta)^2}{\underbrace{7-2\eta+3\eta^2-2+4\eta-2\eta^2}_{5+2\eta+\eta^2}}\right]^{1/2} \therefore \boxed{\sigma_4 = \frac{\sqrt{2}\,(1-\eta)}{\sqrt{5+2\eta+\eta^2}}}$$

Note: When $\eta = 1$, $\sigma_4 = 0$

$\sigma_6 = \frac{1}{\sqrt{2}}$ ($\varphi_6 = 90°$)

This defines $\left\{ \begin{matrix} 4 \\ \tilde{6}[\sqrt{1/2}]_{90°} \end{matrix} \right\} = t\{\tilde{6}, \tilde{4}\}$

Cf. p. 45 $\quad \sigma_6 = \frac{4\eta'-1}{\sqrt{2\eta'^2+2\eta'+2}} \qquad \sigma_4 = \frac{2\eta'}{\sqrt{2\eta'^2-2\eta'+1}}$

These agree!

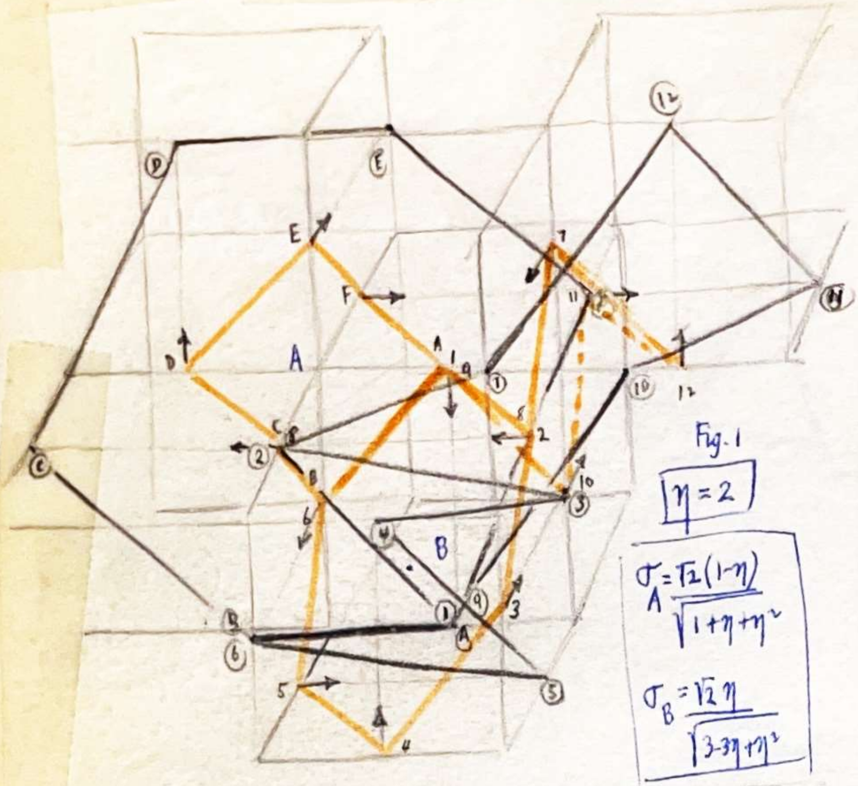


Fig. 1

$\boxed{\eta = 2}$

$$\sigma_A = \frac{\sqrt{2}(1-\eta)}{\sqrt{1+\eta+\eta^2}}$$

$$\sigma_B = \frac{\sqrt{2}\,\eta}{\sqrt{3-3\eta+\eta^2}}$$

These two figures illustrate the effect of the skewing transformation on $\left\{\begin{matrix}\tilde{6}[\sqrt{2}]^{60°}\\ 6\end{matrix}\right\} = t\{\tilde{6}[\sqrt{1/2}], \tilde{6}[\sqrt{2}]\}$.

If the skewing transformation is applied to $\left\{\begin{matrix}\tilde{6}[\sqrt{2}]\\ \tilde{6}\end{matrix}\right\}$, then the equations for σ_A and σ_B become properly anti-symmetric in η. (The equations given here for σ_A and σ_B can be transformed into the antisymmetric ones simply by making the substitution $\zeta = 2\eta - 1$, i.e., $\eta = \frac{\zeta+1}{2}$, in σ_A and σ_B.

Then $\sigma_A(\eta) \to \sigma_A(\zeta) = \frac{\sqrt{2}(1-\eta)}{\sqrt{7+4\eta+\eta^2}}$

and $\sigma_B(\eta) \to \sigma_B(\zeta) = \frac{\sqrt{2}(1+\eta)}{\sqrt{7-4\eta+\eta^2}}$.)

The two pencilled figures in Figs. 1 & 2 are

$$\left\{\begin{matrix}\tilde{6}[-\sqrt{2/7}]^{99.593°}\\ \tilde{6}[\sqrt{8}]_{33.6°}\end{matrix}\right\} = \sigma\Big|_{\eta=2}\left\{\begin{matrix}\tilde{6}[\sqrt{2}]\\ \tilde{6}[0]\end{matrix}\right\} \text{(Fig. 1)}$$

and

$$\left\{\begin{matrix}\tilde{6}[\sqrt{8}]\\ \tilde{6}[-\sqrt{2/7}]\end{matrix}\right\} = \sigma\Big|_{\eta=-1}\left\{\begin{matrix}\tilde{6}[\sqrt{2}]\\ \tilde{6}[0]\end{matrix}\right\} \text{(Fig. 2)}.$$

They are the same polyhedra, but they are symmetrically opposite configurations with respect to the direction of skewing, starting with $\{\tilde{6}, \tilde{4}\}$. Here, they happen to be illustrated in terms of a skewing transformation on the quasi-regular $\left\{\begin{matrix}\tilde{6}[\sqrt{2}]\\ 6\end{matrix}\right\}$, but the most satisfactory general way of describing these skewing transformations is to base them on as symmetrical a starting configuration as possible.

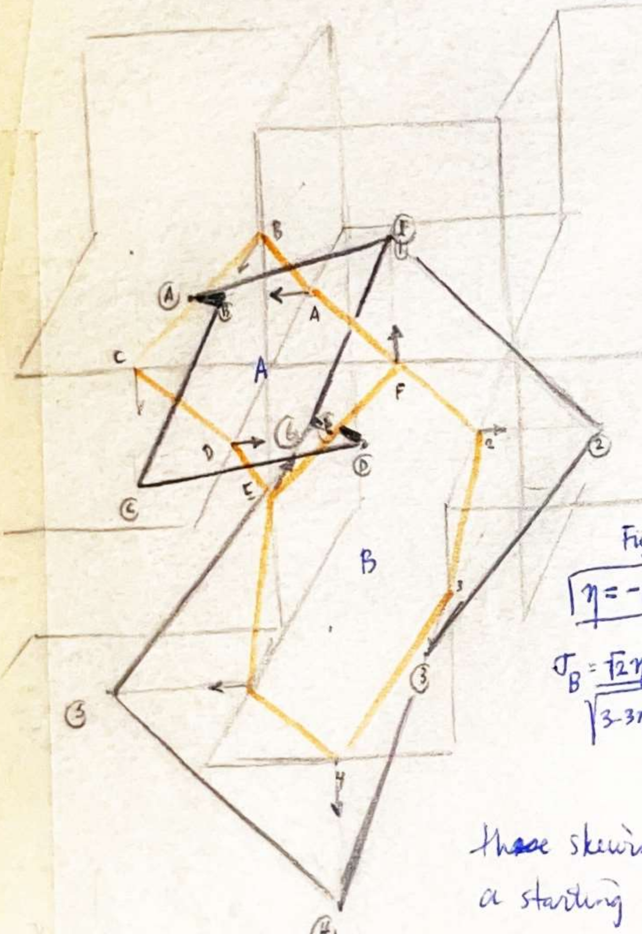
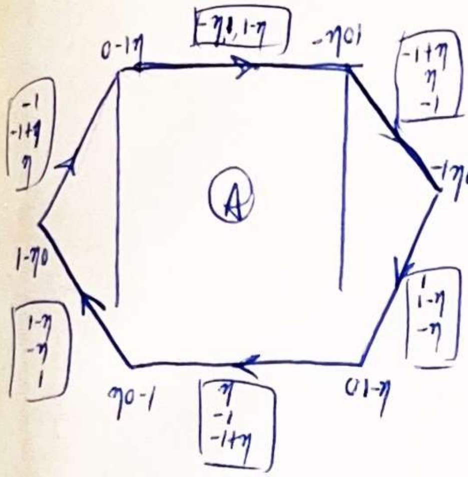


Fig. 2

$\boxed{\eta = -1}$

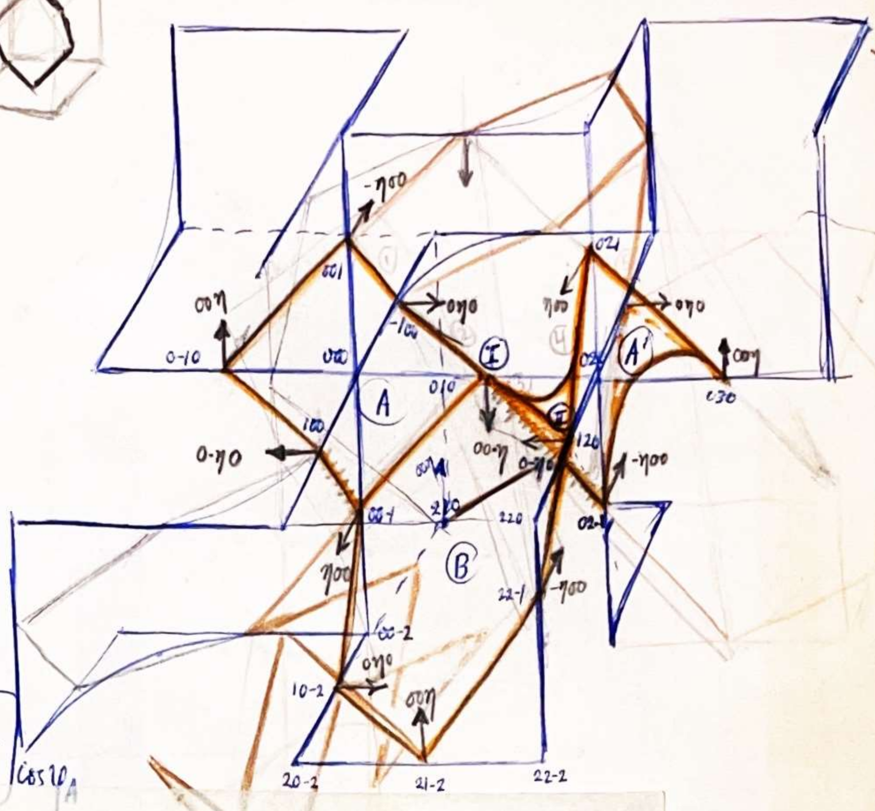
$$\sigma_B = \frac{\sqrt{2}\,\eta}{\sqrt{3-3\eta+\eta^2}}$$

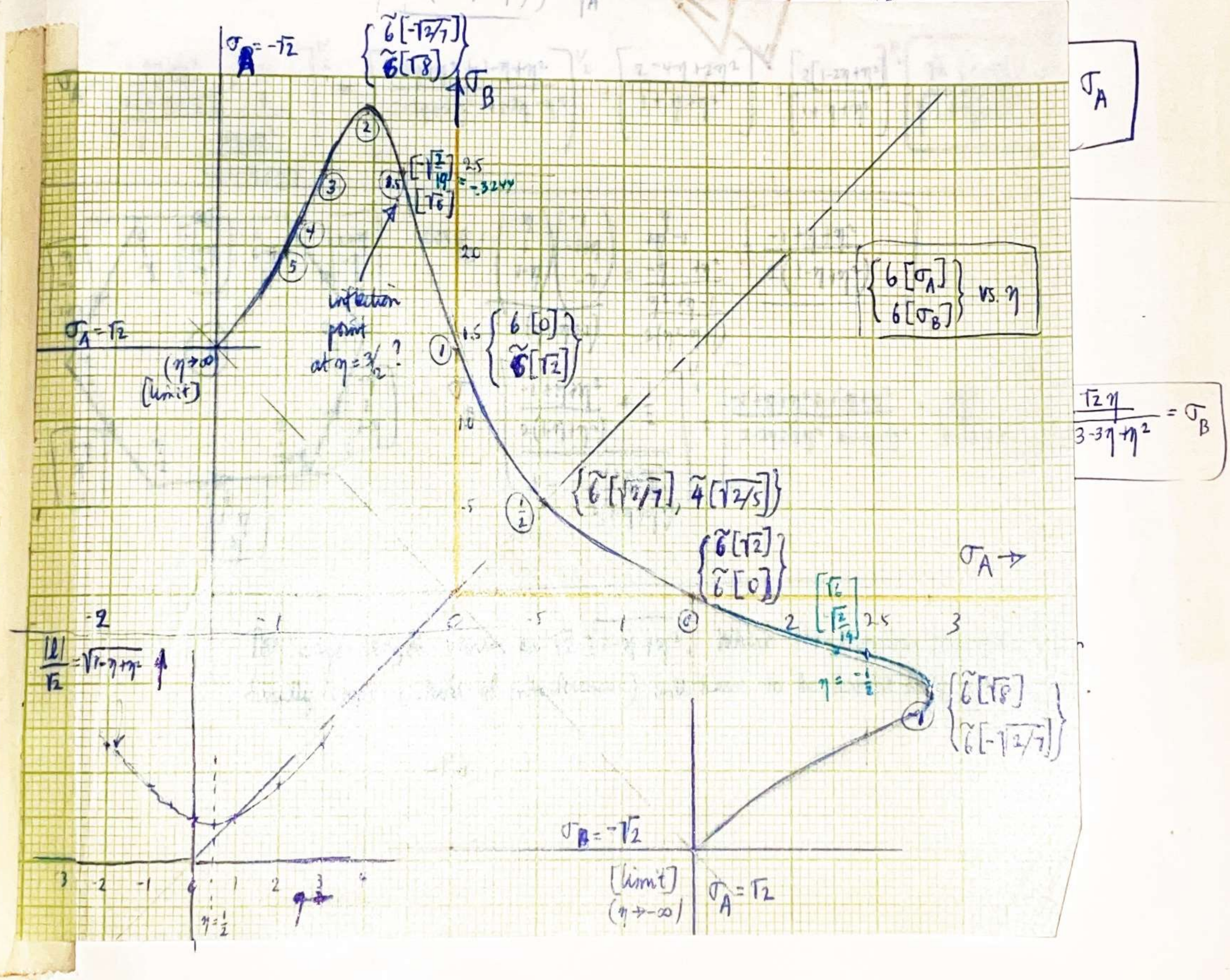
$$\sigma_1 \left\{ \tilde{6}[\sqrt{2}] \right\} = \sigma_1 \left[\left\{ 6, 6 \right\} \right]$$



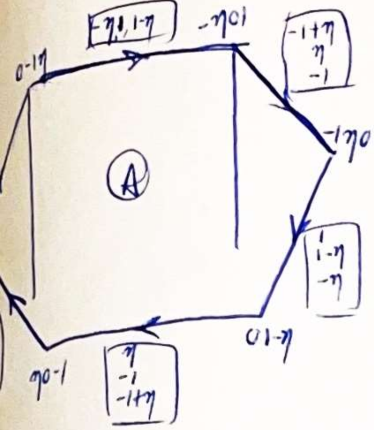
$$\cos\varphi_A = \frac{\begin{pmatrix}\eta\\-1\\-1+\eta\end{pmatrix}\begin{pmatrix}-1+\eta\\ \eta\\ -1\end{pmatrix}}{\eta^2+1+1-2\eta+\eta^2} = \frac{-\eta+\eta^2-\eta-\eta+1}{2-2\eta+2\eta^2} = \frac{\eta^2-3\eta+1}{2(\eta^2-\eta+1)}$$

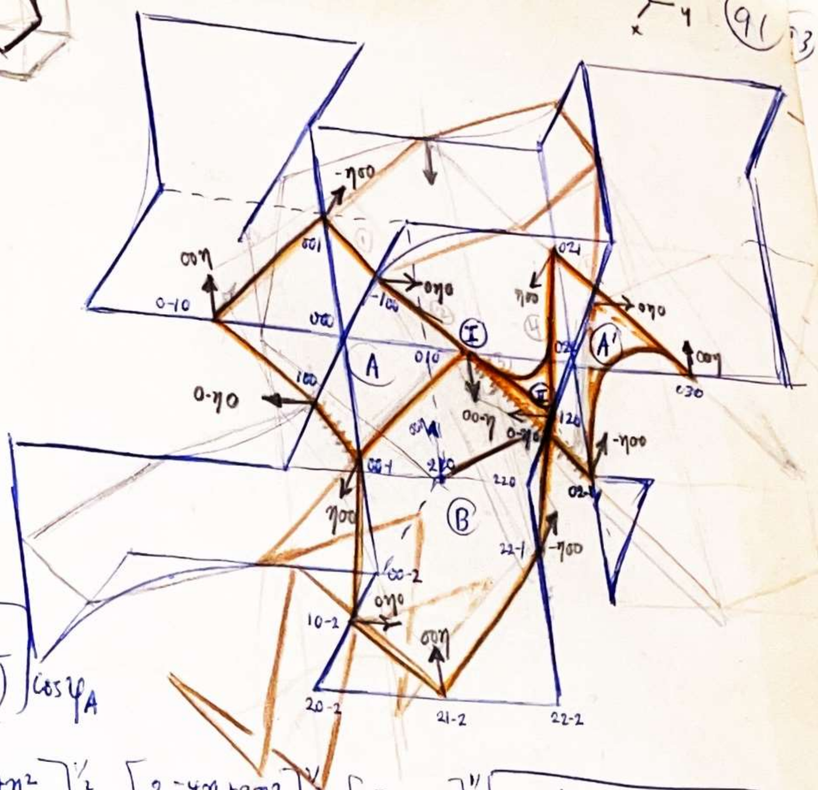
$$\boxed{= \frac{1-3\eta+\eta^2}{2(1-\eta+\eta^2)}} \cos\varphi_A$$





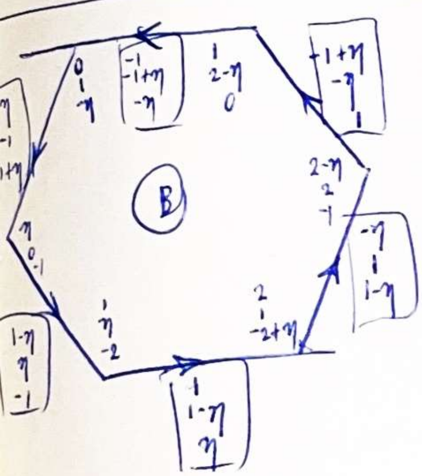
$\sigma_1 \{ \tilde{6}[\sqrt{2}])$





$$\cos\varphi_A = \frac{\begin{pmatrix}\eta\\-1\\-1+\eta\end{pmatrix}\begin{pmatrix}-1+\eta\\\eta\\-1\end{pmatrix}}{\eta^2+1+1-2\eta+\eta^2} = \frac{-\eta+\eta^2-\eta-\eta+1}{2-2\eta+2\eta^2} = \frac{\eta^2-3\eta+1}{2(\eta^2-\eta+1)} = \boxed{\frac{1-3\eta+\eta^2}{2(1-\eta+\eta^2)}}$$

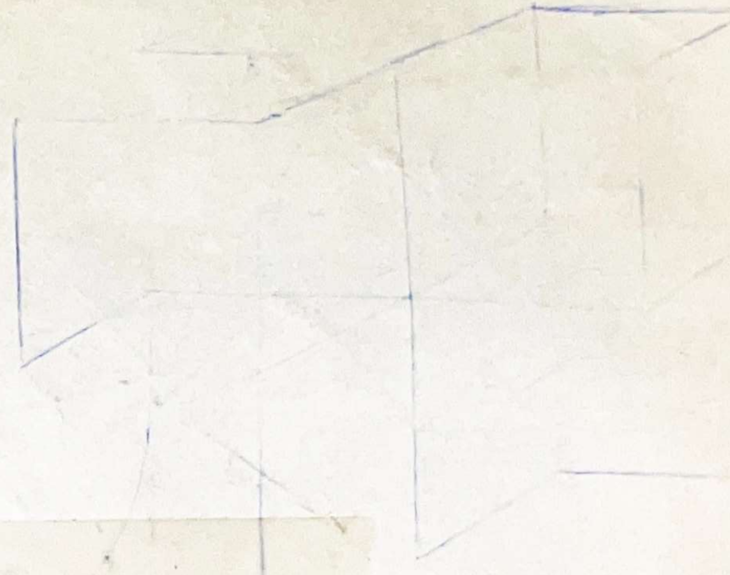
$$\sigma_A = \left[\frac{\frac{1-3\eta+\eta^2}{2(1-\eta+\eta^2)}+\frac{1}{2}}{1-\frac{1-3\eta+\eta^2}{2(1-\eta+\eta^2)}}\right]^{1/2} = \left[\frac{1-3\eta+\eta^2+1-\eta+\eta^2}{2-2\eta+2\eta^2-1+3\eta-\eta^2}\right]^{1/2} = \left[\frac{2-4\eta+2\eta^2}{1+\eta+\eta^2}\right]^{1/2} = \left[\frac{2[1-2\eta+\eta^2]}{1+\eta+\eta^2}\right]^{1/2} = \boxed{\frac{\sqrt{2}(1-\eta)}{\sqrt{1+\eta+\eta^2}} = \sigma_A}$$

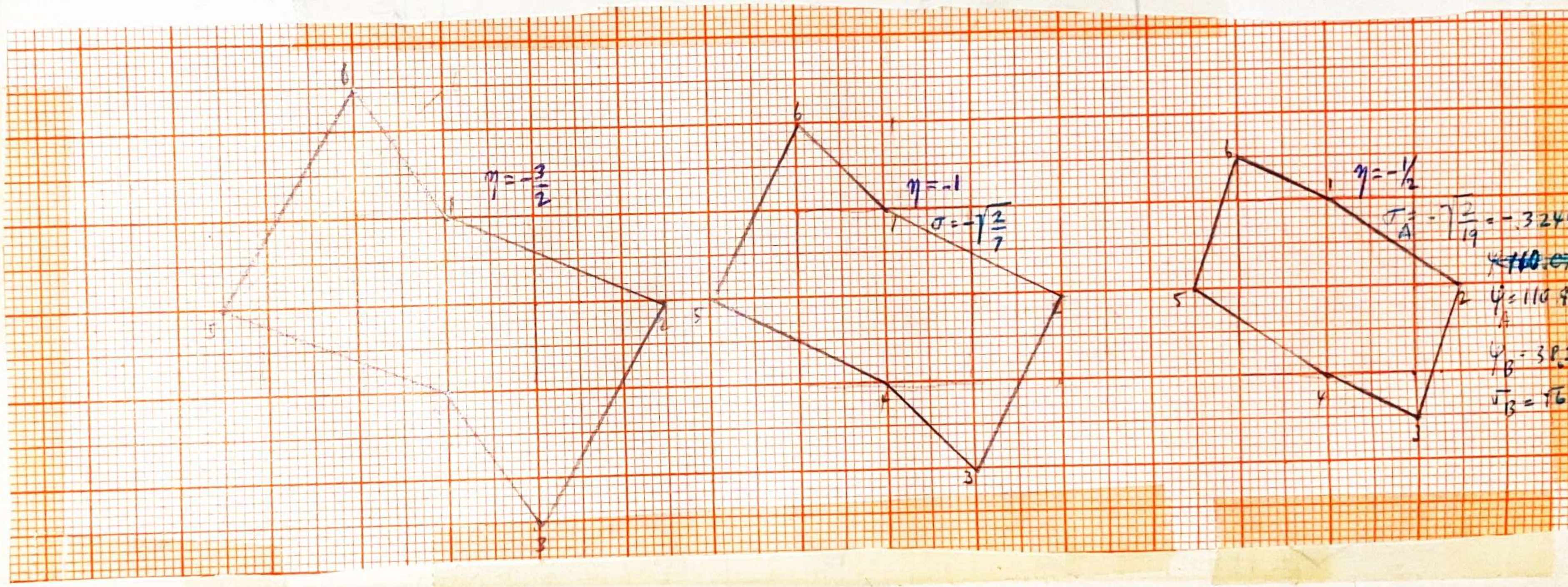


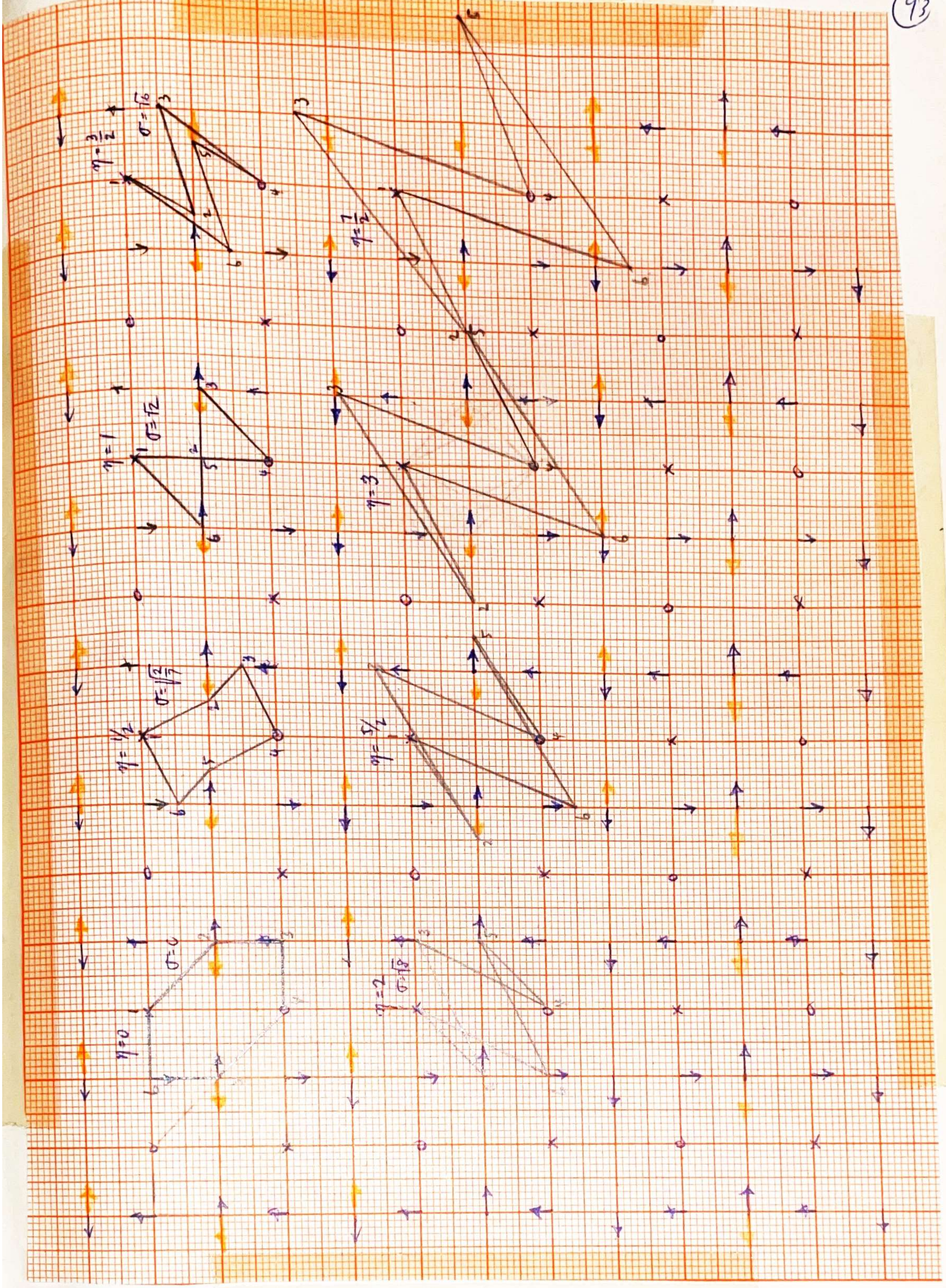
$$\cos\varphi_B = \frac{\begin{pmatrix}-\eta\\1\\1-\eta\end{pmatrix}\begin{pmatrix}-1\\-1+\eta\\-\eta\end{pmatrix}}{2(\eta^2-\eta+1)} = \frac{\eta+\eta-1-\eta+\eta^2}{2(\eta^2-\eta+1)} = \frac{\eta^2+\eta-1}{2(\eta^2-\eta+1)} = \boxed{\frac{-1+\eta+\eta^2}{2(1-\eta+\eta^2)}}$$

$$\sigma_B = \left[\frac{\frac{-1+\eta+\eta^2}{2(1-\eta+\eta^2)}+\frac{1}{2}}{1-\frac{-1+\eta+\eta^2}{2(1-\eta+\eta^2)}}\right]^{1/2} = \left[\frac{-1+\eta+\eta^2+1-\eta+\eta^2}{2-2\eta+2\eta^2+1-\eta-\eta^2} = \frac{2\eta^2}{3-3\eta+\eta^2}\right]^{1/2} = \boxed{\frac{\sqrt{2}\,\eta}{\sqrt{3-3\eta+\eta^2}} = \sigma_B}$$

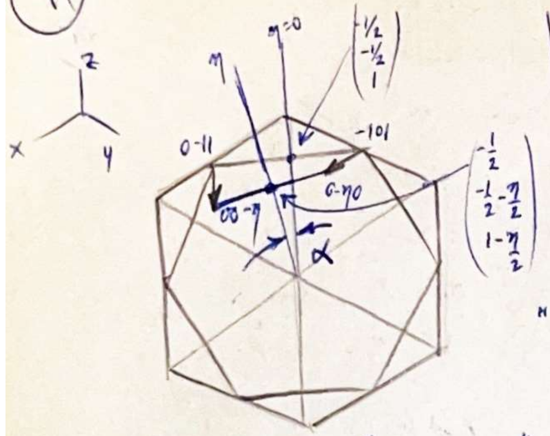
The edge length varies as $\sqrt{2}\sqrt{1-\eta+\eta^2}$. Where the skewing transformation finally stops [short of interferences] will have to be looked into empirically.




6
1
5
2
4
3
η = −3/2
η = −1
σ = −√(2/7)
η = −1/2
σ_A = −√(2/19) = −.3244
~~ψ 110.076~~
ψ_A = 110.924°
ψ_B = 38.213°
σ_B = √6


η = 3/2
σ = √6
η = 1
σ = √2
η = 0
σ = 0
η = 1/2
η = 3
η = 5/2
η = 2
σ = √8

94



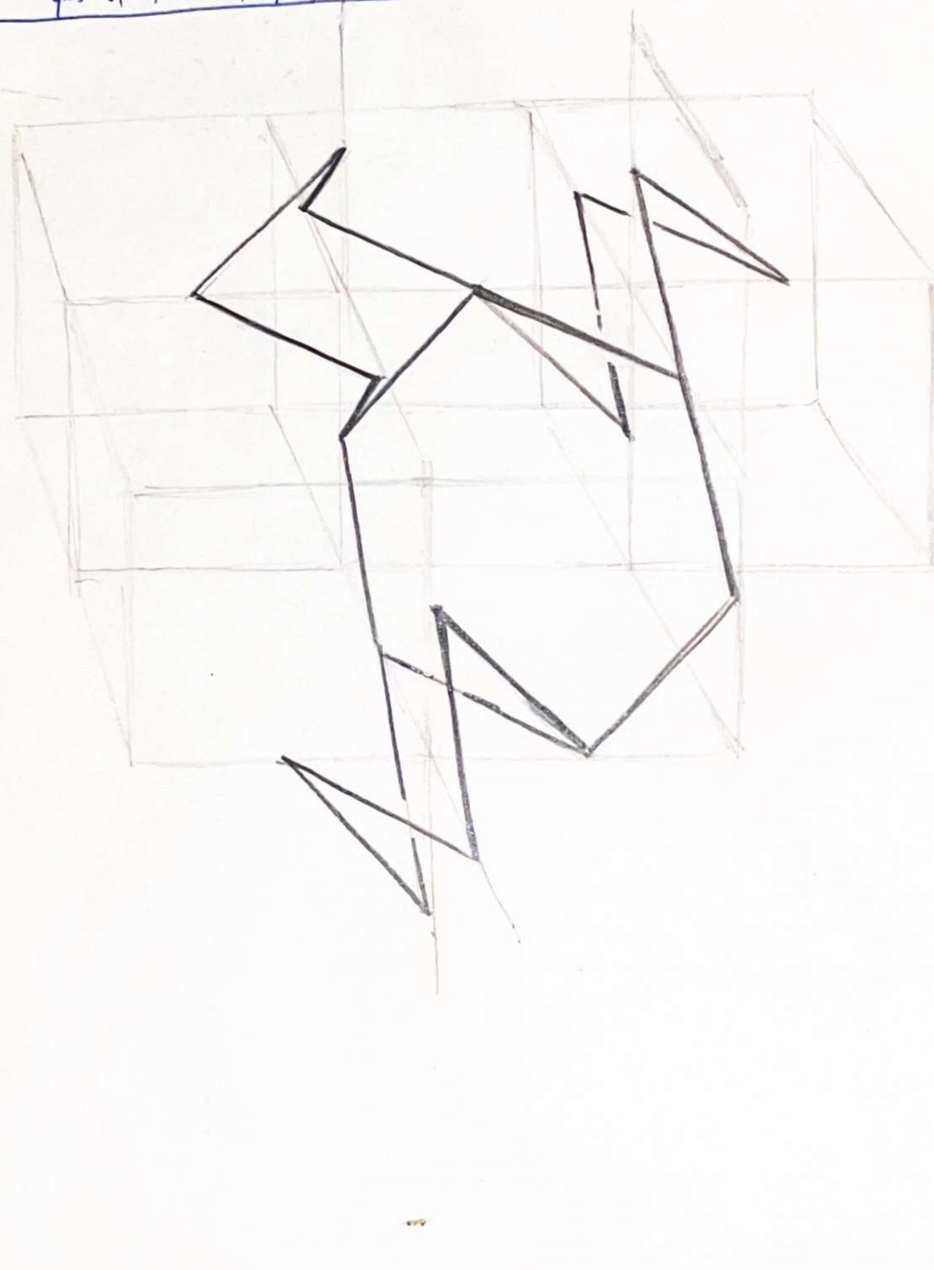
$$\boxed{\cos\alpha = \frac{\begin{pmatrix}-1/2\\-1/2\\1\end{pmatrix}\begin{pmatrix}-1/2\\-1/2-\eta/2\\1-\eta/2\end{pmatrix}}{\sqrt{(\ )(\ )}} = \frac{6-\eta}{6\sqrt{1-\eta/3+\eta^2/3}}}$$

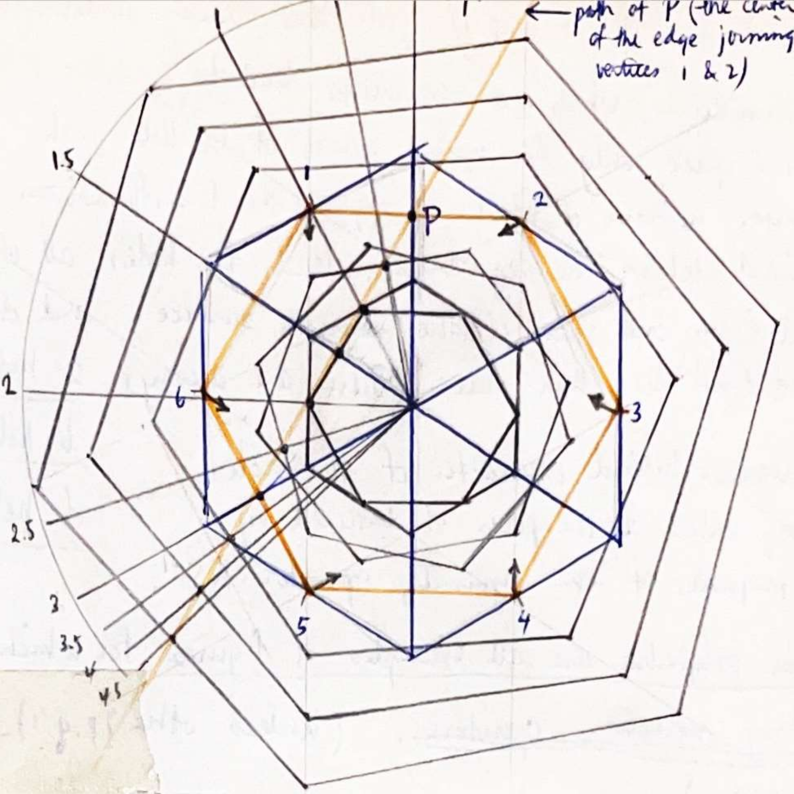
I shall call these skewing transformations "TWISTING transformations", since they rotate the polygons as well as change their skewness.

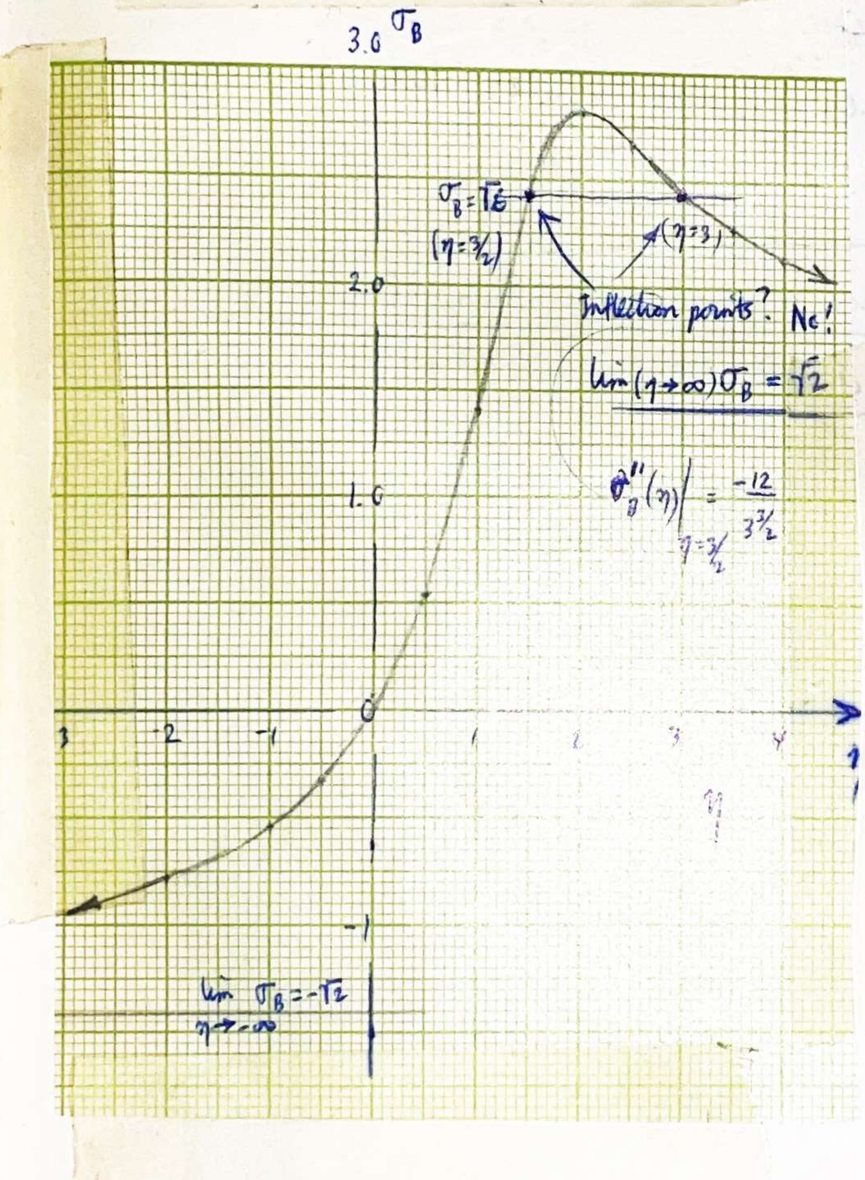
("TWISTING" is more euphonious than "skewing".)

(The vertex figure of any vertex sweeps out a path in 3-space, but the ends of the vertex figures, i.e., the centers of the edges of the hexagon, follow linear paths.

The edges of $\left\{ \begin{matrix} 6 \\ \tilde{6} \end{matrix} \right\}$ form ~~a the~~ an "e.c.c." (edge-centered cubic) array of vertices, joined in a graph of degree 4.







$$\boxed{\sigma_B = \frac{\sqrt{2}\,\eta}{\sqrt{3 - 3\eta + \eta^2}}}$$

MINIMUM PROJECTED AREA

η	σ_B	
0	0	0
.5	$\sqrt{2/7}$	.5345
1	$\sqrt{2}$	1.414
1.5	$\sqrt{6}$	2.449
2.0	$\sqrt{8}$	2.828
2.5	$5\sqrt{2/7}$	2.6725
3.0	$\sqrt{6}$	2.449
3.5	$7\sqrt{2/19}$	2.272
4.0	$4\sqrt{2/7}$	2.138
4.5		
∞	$\sqrt{2}$	1.414
-.5	$-\sqrt{2/19}$	-.3246
-1	$-\sqrt{2/7}$	-.5345
-2	$-2\sqrt{2/13}$	-.7848
-3	$-3\sqrt{2/21}$	
-4	$-4\sqrt{\frac{2}{31}}$	
$-\infty$	$-\sqrt{2}$	-1.414

(96) (cont. from p. 95 [opp.]) of the faces, vertex figures, and holes. These transformations, which are continuous, lead to a finite # of faces, etc. in any finite region of space only for special values of the hole pitch. The hole pitch is a convenient parameter in terms of which to express the transformations, starting with the $\{p, \tilde{q} \mid n(c)\}$ as initial states. The reg. transf. leads to holes all of the same handedness when viewed from one side of the labyrinth surface, and of the opposite handedness when viewed from the other side. There are always

2 holes at each vertex in a $\{6,4)$ (1 lh & 1 rh)
6 holes " " " $\{4,6\}$ (3 lh & 3 rh
6 holes " " " $\{6,6\}$ (3 lh & 3 rh

The regular helical character of the holes for any value of the pitch is insured by the properties of the symmetry operator RS^{-1}.

These polyhedra are all examples of figures for which both R and S are rotary reflection operators. (Unless other $\{p.q\text{'s}\}$ exist, these 9 exhaust the possibilities.)

Make plastic minimal surfaces on the regular helical polygons (analogs of the helicoid) ☆.

Slope of the circum-helix = axial advance per unit circumferential length

Use this slope $= \tau$ as the measure of the regular helical polygons $\{n(\tau)\}$