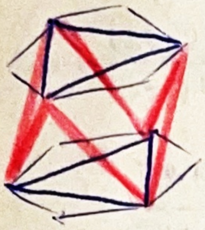
6/13/68 Friday

$\begin{Bmatrix}\tilde{6}\\ \frac{\widetilde{10}}{3}\end{Bmatrix}$ This is related to the icosidodecahedron in the same way $\begin{Bmatrix}\tilde{6}\\ \tilde{4}\end{Bmatrix}$ is related to the cuboctahedron:

<u>Each pair of parallel [and opposite] triangle faces have their vertices joined to form a regular skew hexagon.</u>

(There is only one way to do this.)



These edges also define the other regular faces — $\begin{Bmatrix}\frac{\widetilde{10}}{3}\end{Bmatrix}$ for $\begin{Bmatrix}3\\ 5\end{Bmatrix}$ and $\{\tilde{4}\}$ for $\begin{Bmatrix}3\\ 4\end{Bmatrix}$.

$\begin{Bmatrix}\tilde{6}\\ \frac{\widetilde{10}}{3}\end{Bmatrix}$ is not isomorphic to any other known polyhedron, whereas $\begin{Bmatrix}\tilde{6}\\ \tilde{4}\end{Bmatrix}$ <u>is</u> isomorphic to $\begin{Bmatrix}6\\ 4\end{Bmatrix}$.

FALSE! 6/14/68

~~$\begin{Bmatrix}5\\ \frac{5}{2}\end{Bmatrix}$ and $\begin{Bmatrix}3\\ \frac{5}{2}\end{Bmatrix}$ do not appear to generate any quasi-regular saddle polyhedra.~~

When two reciprocal tetrahedra $\{3,3\}$ are superimposed, giving rise to $\{3,4\}$ as the quasi-regular polyhedron, and $\{3,4\}$ is then subjected to the same operation, the resulting $\begin{Bmatrix}\tilde{p}\\ \tilde{q}\end{Bmatrix}$ is $\begin{Bmatrix}\tilde{6}\\ \tilde{6}\end{Bmatrix} \equiv \{\tilde{6}, 4\}$.

Thus, $\{\tilde{6},4\}$ can be treated as a quasi-regular saddle polyhedron, in somewhat the same sense as can the regular octahedron $\{3,4\}$.

In $\begin{Bmatrix}\tilde{6}\\ \frac{\widetilde{10}}{3}\end{Bmatrix}$, just as in the $q\{\tilde{h},q\}_p$, each $\{\tilde{6}\}$ face shares 2 <u>opposite</u> [parallel] edges with each of the [3] adjoining $\begin{Bmatrix}\frac{\widetilde{10}}{3}\end{Bmatrix}$ faces, and each $\begin{Bmatrix}\frac{\widetilde{10}}{3}\end{Bmatrix}$ face shares 2 <u>opposite</u> (parallel) edges with each of the 5 adjoining $\{\tilde{6}\}$ faces. In contrast to this, in $\begin{Bmatrix}\tilde{6}\\ \tilde{4}\end{Bmatrix}$, each $\{\tilde{6}\}$ face shares only one edge with each of the 6 $\{\tilde{4}\}$ neighbors, and each $\{\tilde{4}\}$ face shares only one edge with each of the 4 $\{\tilde{6}\}$ neighbors.

$M = \frac{1}{2}(-1, -1-\underset{-\tau}{\tau^{-1}}, 1+\tau)$ $\quad \vec{JM} = \frac{1}{2}[-2-\tau, -1-\tau, 1+2\tau]$ $\quad |\vec{JM}| = \sqrt{3+4\tau}$ ✓

$N = \frac{1}{2}(-1-\tau^{-1}, -1-\tau, 1)$ $\quad \vec{JN} = \cancel{\frac{1}{2}[1, -\tau, -1-\tau]}$

$= \frac{1}{2}[-1-2\tau, -2-\tau, 1+\tau]$ $\quad |\vec{JN}| = \sqrt{3+4\tau}$

$J = \frac{1}{2}(1+\tau, 1, \underset{-\tau}{-1-\tau^{-1}})$

$$\cos\varphi_{\cancel{6}} = \frac{JM \cdot JN}{|JM||JN|} = \frac{\frac{1}{4}\begin{pmatrix}-(2+\tau)\\-(1+\tau)\\(1+2\tau)\end{pmatrix}\begin{pmatrix}-(1+2\tau)\\-(2+\tau)\\(1+\tau)\end{pmatrix}\begin{pmatrix}4+7\tau\\+3+4\tau\\+3+5\tau\end{pmatrix}}{3+4\tau} = \frac{\frac{1}{4}[10+16\tau]}{3+4\tau} = \frac{\frac{1}{2}[5+8\tau]}{3+4\tau}$$

$$= \frac{\frac{5}{2}+4\tau}{3+4\tau}$$

$\sqrt{5} = 2.236067977$
6.708203931
8.291796069

$$\cos\varphi_{\frac{10}{3}} = \frac{\vec{JM}\cdot\vec{JL}}{3+4\tau} = \frac{\frac{1}{2}\begin{pmatrix}-(2+\tau)\\-(1+\tau)\\1+2\tau\end{pmatrix}\begin{pmatrix}-(1+\tau)\\0\\\tau\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3+4\tau\\+0\\+3+2\tau\end{pmatrix}}{3+4\tau} = \frac{\frac{1}{2}(6+6\tau)}{3+4\tau} = \frac{3+3\tau}{3+4\tau} = \frac{1}{5}[-3\tau+9]$$

$$= \frac{15-3\sqrt{5}}{10} = .8291796067$$

$\sin^{-1}(.9472135956) = 71.30059294°$ $\quad 18.69940706$ $\quad \therefore \cos^{-1}(.947\ldots) = 18.69940706°$ ✓

$\sin^{-1}(.8291796067) = 56.01455578°$ $\quad 33.98544422$ $\quad \therefore \cos^{-1}(.8291\ldots) = 33.98544422°$ ✓

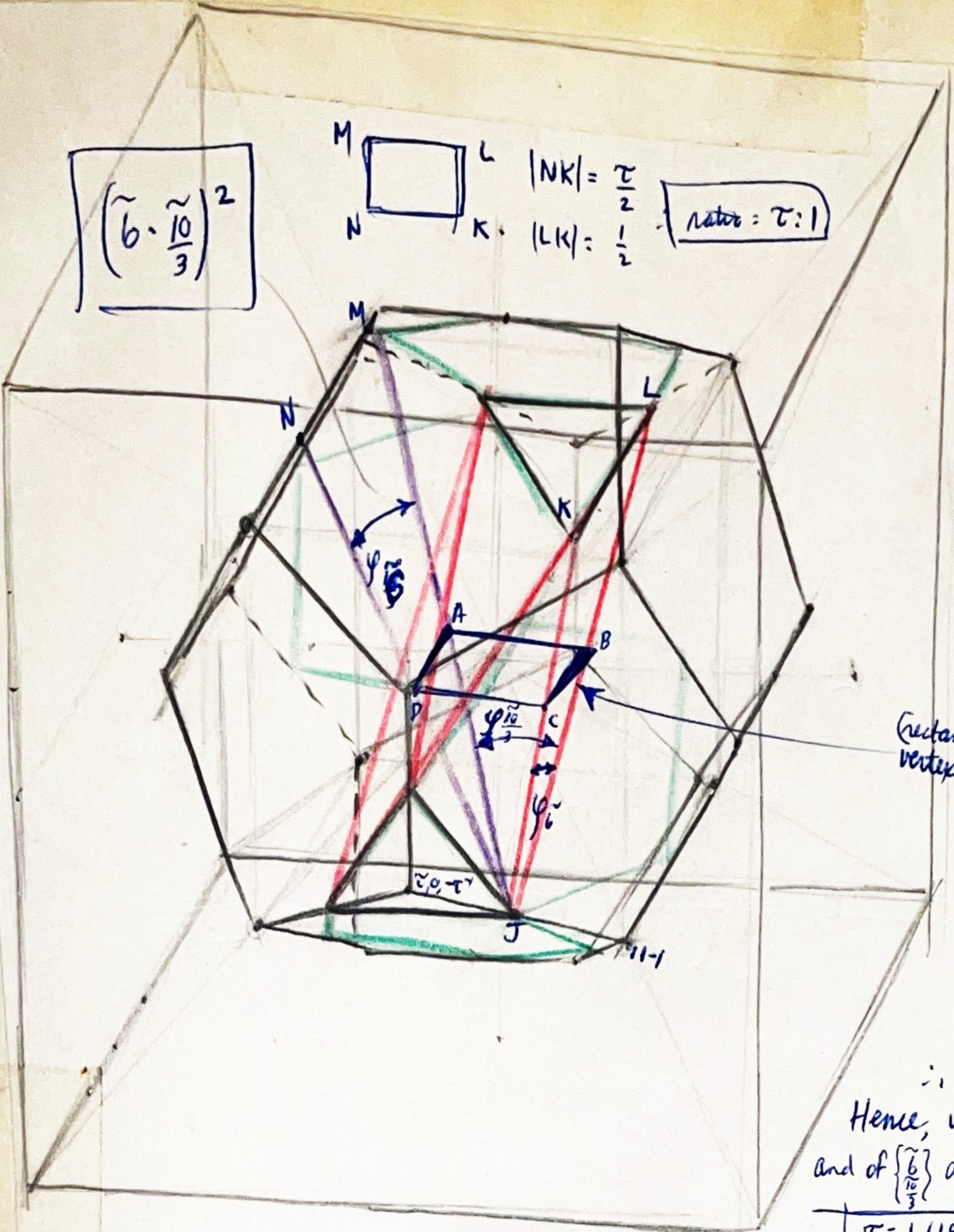
$$\therefore \boxed{\begin{aligned}\varphi_{\frac{10}{3}} &= 33.98544422°\\ \varphi_6 &= 18.69940706°\end{aligned}}$$

$\cos\varphi_{\frac{10}{3}} = \frac{15-3\sqrt{5}}{10}$ $\quad \sigma_{\frac{10}{3}} = \left(\frac{1+5\sqrt{5}}{4}\right)^{1/2} = 1.745017184$

$\cos\varphi_6 = \frac{5+2\sqrt{5}}{10}$ $\quad \sigma_6 = (14+6\sqrt{5})^{1/2} = 5.236067977$

$$\boxed{\left(\tilde{6}\cdot\tfrac{\tilde{10}}{3}\right)^2}$$

M ▭ L, N, K · $|NK| = \frac{\tau}{2}$, $|LK| = \frac{1}{2}$ · $\boxed{\text{ratio} : \tau : 1}$



$J = \frac{1}{2}(1+\tau, 1, -\tau)$

$K = (-\tau, 0, 0)$

$L = \frac{1}{2}(-1-\tau, 1, \tau)$

$M = \frac{1}{2}(-1, -\tau, 1+\tau)$

$N = \frac{1}{2}(-\tau, -1-\tau, 1)$

$A = \frac{1}{4}[\tau, +1-\tau, 1]$

$B = [0, \tfrac{1}{2}, 0]$

$C = \frac{1}{4}[1-\tau, 1, -\tau]$

$D = \frac{1}{4}[1, -\tau, 1-\tau]$

(rectangular) vertex figure

$\vec{AB} = \frac{1}{4}[-\tau, 1+\tau, -1]$

$\vec{BC} = \frac{1}{4}[1-\tau, -1, -\tau]$

$|AB| = \frac{\tau}{2} = .80901\ldots$

$|\vec{BC}| = \frac{1}{2}$

$\therefore$ ratio $= \tau : 1$

Hence, vertex figures of $\left\{\begin{smallmatrix}3\\5\end{smallmatrix}\right\}$ and of $\left\{\begin{smallmatrix}\tilde{6}\\ \frac{\tilde{10}}{3}\end{smallmatrix}\right\}$ are geometrically SIMILAR!

$$\tau = 1.618033989 = 2\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{2}$$

$J = \left(\frac{1+\tau}{2}, \frac{1}{2}, \frac{-1-\tau^{-1}}{2}\right)$

$K = (-\tau, 0, 0)$

$L = \left(\frac{-1-\tau}{2}, \frac{1}{2}, \frac{1+\tau^{-1}}{2}\right)$

$\vec{JK} = \left(\frac{-3\tau-1}{2}, -\frac{1}{2}, \frac{1+\tau^{-1}}{2}\right) = \frac{1}{2}\left[-(1+3\tau), -1, 1+\tau^{-1}\right] = \frac{1}{2}[-1-3\tau, -1, \tau]$ ✓

$\vec{JL} = \left(-1-\tau, 0, \underbrace{1+\tau^{-1}}_{\tau}\right)$ ✓

$$\cos\varphi_6 = \frac{\vec{JK}\cdot\vec{JL}}{|\vec{JK}||\vec{JL}|} = \frac{\frac{1}{2}\left[1+4\tau+3\tau^2+1+2\tau^{-1}+\tau^{-2}\right]}{\sqrt{\frac{1}{4}\left[1+6\tau+9\tau^2\right]+1+1+2\tau^{-1}+\tau^{-2}}\sqrt{1+2\tau+\tau^2+1+2\tau^{-1}+\tau^{-2}}}$$

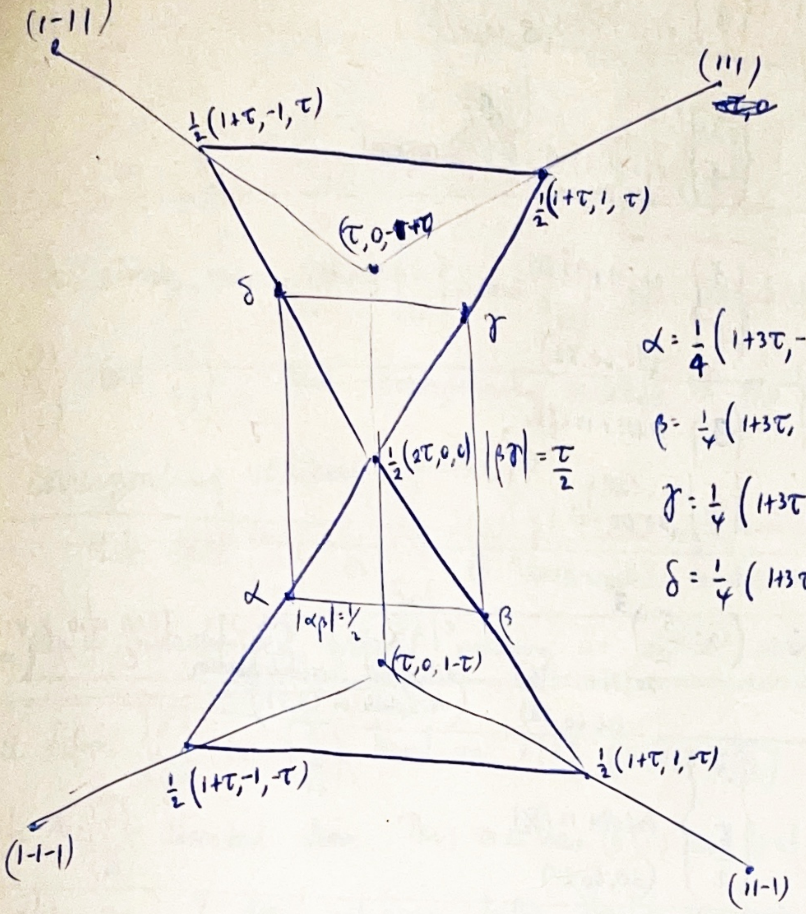
$$= \frac{\frac{1}{2}\left[1+4\tau+3(\tau+1)+1+2(\tau-1)+\left\{\overset{\tau^2-2\tau+1}{(\tau-1)^2} = -\tau+2\right\}\right] = \frac{1}{2}(5+8\tau)}{\frac{1}{2}\sqrt{(12+16\tau)(3+4\tau)}} = \boxed{\frac{\frac{1}{2}(5+8\tau)}{3+4\tau}} = \frac{8.97212}{9.47212} = .9472$$

$$= \frac{2.5+4\tau}{3+4\tau} = \frac{1}{10}(3+4\tau) = .9472135956 = \left(\frac{5+2\sqrt{5}}{10}\right)$$ ✓

$$\boxed{\varphi_6 = 18.755^\circ}$$

cont B

Intermediate Surface Theory



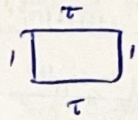
$\alpha = \frac{1}{4}(1+3\tau,-1,-\tau)$ $\quad \vec{\alpha\beta} = \frac{1}{4}(0,2,0)$ $\quad |\alpha\beta| = \frac{1}{2}$

$\beta = \frac{1}{4}(1+3\tau,1,-\tau)$ $\quad \vec{\beta\gamma} = \frac{1}{4}(0,0,2\tau)$ $\quad |\beta\gamma| = \frac{\tau}{2}$

$\gamma = \frac{1}{4}(1+3\tau,1,\tau)$

$\delta = \frac{1}{4}(1+3\tau,-1,\tau)$

$$\boxed{\frac{|\beta\gamma|}{|\alpha\beta|} = \tau : 1}$$



Thus, vertex figures of $\begin{Bmatrix}\tilde{6}\\ \tilde{10}\\ 3\end{Bmatrix}$ and $\begin{Bmatrix}3\\ \\ 5\end{Bmatrix}$ are SIMILAR!

(Cf. SIMILAR vertex figures of $\begin{Bmatrix}\tilde{6}\\ \tilde{4}\end{Bmatrix}$ and $\begin{Bmatrix}6\\ 4\end{Bmatrix}$.)

The tetratrihedron $\frac{3}{2}\,3 \mid 2$ does not give rise to a $\left\{\begin{matrix} p \\ q \end{matrix}\right\}$
Neither do the other 8 q'q'lu

So far, we have

$\{\tilde{6}, 4\}_3 \longleftrightarrow \left\{\begin{matrix} \tilde{6} \\ \tilde{6} \end{matrix}\right\} : \left\{\begin{matrix} 3 \\ 3 \end{matrix}\right\} \longleftrightarrow \{3, 4\}_6$

$2\{\tilde{6}\} + 2\{\tilde{6}\}$ $(6, 12, 4)$; $4\{3\} + 4\{3\}$ $(6, 12, 8)$

$\begin{pmatrix} 6 \\ 4 \end{pmatrix}$ $\cong$ (isomorphic) $4\{\tilde{6}\} + 6\{\tilde{4}\}$ $\left\{\begin{matrix} \tilde{6} \\ \tilde{4} \end{matrix}\right\} : \left\{\begin{matrix} 3 \\ 4 \end{matrix}\right\}$ $8\{3\} + 6\{4\}$ $(12, 24, 14)$ | $\cancel{\left\{\begin{matrix} 8 \\ 6 \end{matrix}\right\}}$ $\cancel{8\{3\}+6\{4\}}$

$(12, 24, 10)$

Aside from isomorphisms within the set, No isomorphs known.

$10\{\tilde{6}\} + 6\{\tfrac{\tilde{10}}{3}\}$ $\left\{\begin{matrix} \tilde{6} \\ \frac{\tilde{10}}{3} \end{matrix}\right\} : \left\{\begin{matrix} 3 \\ 5 \end{matrix}\right\}$ $20\{3\} + 12\{5\}$

$(30, 60, 16)$; $(30, 60, 32)$

" $10\{\tilde{6}\} + 6\{\tilde{10}\}$ $\left\{\begin{matrix} \tilde{6} \\ \tilde{10} \end{matrix}\right\} : \left\{\begin{matrix} 3 \\ \frac{5}{2} \end{matrix}\right\}$ $20\{3\} + 12\{\tfrac{5}{2}\}$

$(30, 60, 16)$; $(30, 60, 32)$

" $10\{\tilde{6}\} + 6\{\tilde{10}\}$ $\left\{\begin{matrix} \tilde{6} \\ \tilde{10} \end{matrix}\right\}' : \left(3 \cdot \frac{5}{2}\right)^3$ $20\{3\} + 12\{5/2\}$ $(20, 60, 32)$; $3 \mid 3\,\frac{5}{2}$ Fig. 39 ditrigonal icosidodecahedron (inscriptable in $(5,3)$) (see also p. 413 C, L-H, & M)

$(20, 60, 16)$

" $6\{\tilde{10}\} + 6\{\tfrac{\tilde{10}}{3}\}$ $\left\{\begin{matrix} \tilde{10} \\ \frac{\tilde{10}}{3} \end{matrix}\right\} : \left\{\begin{matrix} 5 \\ \frac{5}{2} \end{matrix}\right\}$ $12\{5\} + 12\{\tfrac{5}{2}\}$ $(30, 60, 24)$

$(30, 60, 12)$

" $10\{\tilde{6}\} + 6\{\tfrac{\tilde{10}}{3}\}$ $\left\{\begin{matrix} \tilde{6} \\ \frac{\tilde{10}}{3} \end{matrix}\right\} : (3 \cdot 5)^3$ $20\{3\} + 12\{5\}$ $(20, 60, 32)$ $\longleftrightarrow$ $\frac{3}{2} \mid 3\,5$ Fig. 61 ditrigonal icosidodecahedron (inscriptable in $(5,3)$) (cf. p. 413)

$(20, 60, 16)$

" $6\{\tilde{10}\} + 6\{\tfrac{\tilde{10}}{3}\}$ $\left\{\begin{matrix} \tilde{10} \\ \frac{\tilde{10}}{3} \end{matrix}\right\} : \left(5 \cdot \frac{5}{2}\right)^3$ $(20, 60, 24)$; $3 \mid \frac{5}{3}\,5$ Fig. 53 (ditrigonal icosidodecahedron (inscriptable in $(5,3)$)) (cf. p. 413)

$12\{5\} + 12\{\tfrac{5}{2}\}$

$(20, 60, 12)$

Thus, there is one $\left\{\begin{matrix} \tilde{p} \\ \tilde{q} \end{matrix}\right\}$ corresponding to every $\left\{\begin{matrix} p \\ q \end{matrix}\right\}$ except those which ~~have~~ include equatorial polygon faces.

Saturday, June 15, 1968

hah!

The fourth — ~~and final~~ one — of the $\begin{Bmatrix}\tilde{p}\\ \tilde{q}\end{Bmatrix}$ [finite] is $\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix}$.

$\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix}$ is isomorphic to $\begin{Bmatrix}\tilde{6}\\ \frac{\tilde{10}}{3}\end{Bmatrix}$. In fact, $\boxed{\begin{aligned}&\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix} : \begin{Bmatrix}\frac{5}{2}\\ 3\end{Bmatrix}\\ = &\begin{Bmatrix}\tilde{6}\\ \frac{\tilde{10}}{3}\end{Bmatrix} : \begin{Bmatrix}5\\ 3\end{Bmatrix}\end{aligned}}$

This simply means that $\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix}$ and $\begin{Bmatrix}\tilde{6}\\ \frac{\tilde{10}}{3}\end{Bmatrix}$ are inscribed in the isomorphic $\begin{Bmatrix}p\\ q\end{Bmatrix}$'s, $\begin{Bmatrix}\frac{5}{2}\\ 3\end{Bmatrix}$ and $\begin{Bmatrix}5\\ 3\end{Bmatrix}$, and corresponding vertices of the $\begin{Bmatrix}\tilde{p}\\ \tilde{q}\end{Bmatrix}$ pair are also corresponding vertices of the $\begin{Bmatrix}p\\ q\end{Bmatrix}$ pair.

Note that $\begin{Bmatrix}5/2\\ 5\end{Bmatrix}$ or $\begin{Bmatrix}5\\ 5/2\end{Bmatrix}$ is "automorphic" in the sense that its pentagons and pentagrams can be interchanged without altering its abstract description. [Coxeter, Reg. Polytopes, p. 107]

A fifth $\begin{Bmatrix}\tilde{p}\\ \tilde{q}\end{Bmatrix}$ is $\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix}$, based on $3|3\,\frac{5}{2}$ $\left(20\{3\} + 12\{\frac{5}{2}\}\right)$. This is not geometrically similar to the $\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix}$ described above. This one has $\{\tilde{6}\}$'s and $\{\tilde{10}\}$'s of different skewness from the other one. I don't yet know whether they're isomorphic. This one has $10\{\tilde{6}\} + 6\{\tilde{10}\}$, also.

The recipe for this one — let's temporarily call it $\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix}'$ — is as follows:

Inscribe a ~~xx~~ $\{\tilde{6}\}$ between each pair of opposite parallel Δ's of $3|3\,\frac{5}{2}$.

$3|3\,5/2$ is the figure obtained by inscribing a $\begin{Bmatrix}5\\ 2\end{Bmatrix}$ in each face of $\begin{Bmatrix}5\\ 3\end{Bmatrix}$.

This leaves 20 $\{3\}$'s which intersect each other.

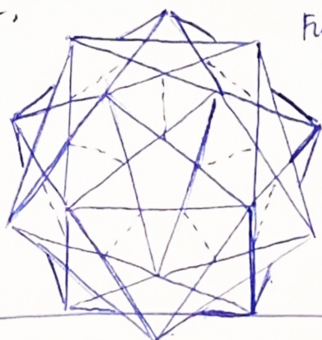


Fig. 39 of Coxeter Longuet-Higgins & Miller

$3|3\,\frac{5}{2}$

$\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix}'$ is almost certainly not isomorphic to $\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix} : \left(3 \cdot \frac{5}{2}\right)^2$

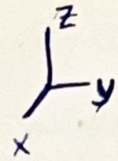
$\left(\begin{Bmatrix}\tilde{6}\\ \tilde{10}\end{Bmatrix}' : \left(3 \cdot \frac{5}{2}\right)^3 \right.$

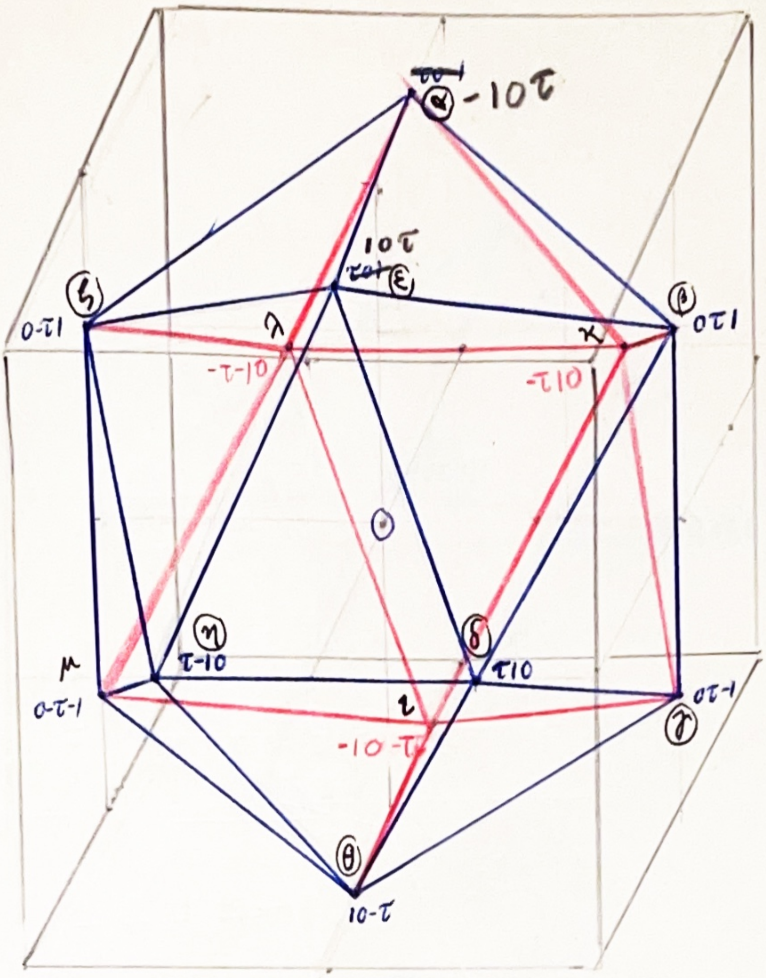
A sixth is $\begin{Bmatrix}\tilde{10}\\ \frac{\tilde{10}}{3}\end{Bmatrix} : \begin{Bmatrix}5\\ \frac{5}{2}\end{Bmatrix}$

$\{\tilde{10}\}$ comes from opposite $\begin{Bmatrix}5\\ 2\end{Bmatrix}$'s

$\begin{Bmatrix}\frac{\tilde{10}}{3}\end{Bmatrix}$ comes from " $\{5\}$'s

A seventh





Length of each edge = 2

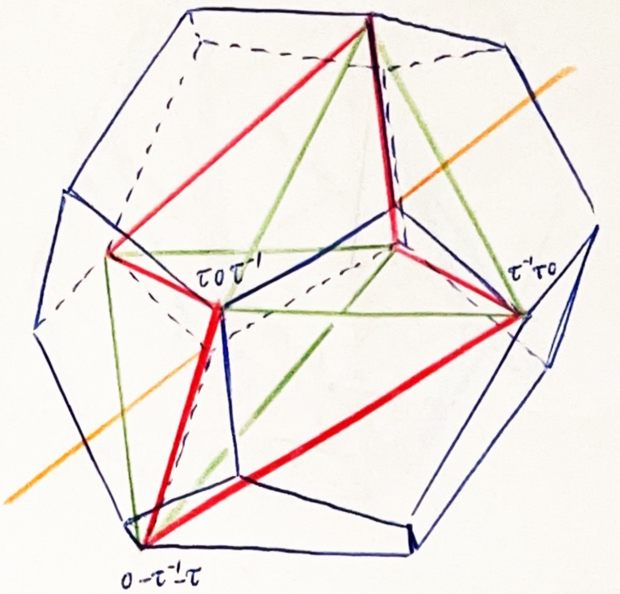
Radius of circumsphere = $\sqrt{2+\tau} \cong 1.902113033$

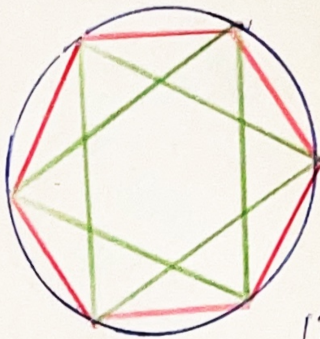
$20\{3\} + 12\{\tfrac{5}{2}\}$

$\{3\cdot\tfrac{5}{2}\}^3$

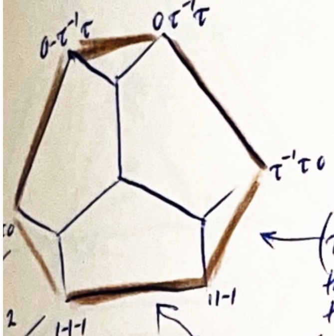
✓ $\left(\widetilde{6}\cdot\widetilde{10}\right)^3$ $\quad 10\{\widetilde{6}\} + 6\{\widetilde{10}\}$

$\{\widetilde{6}\}$





$\{\widetilde{6}\}$

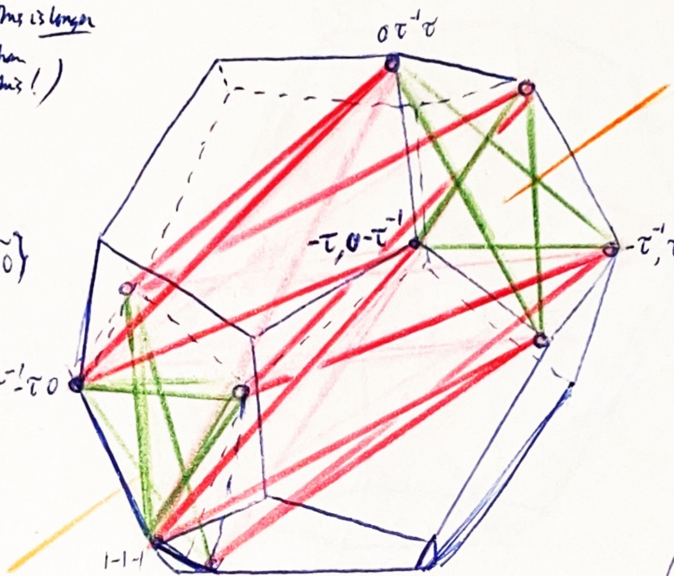


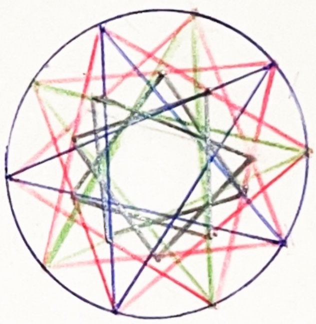
(This is longer than this!)

$|\ell| = 2\tau^{-1}$

vertex fig. is similar to that of $(3\cdot\tfrac{5}{2})^3$

$\{\widetilde{10}\}$

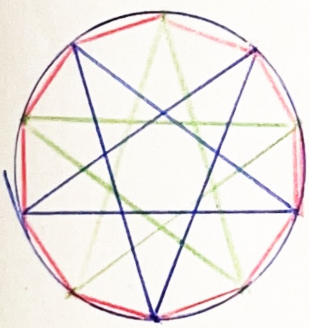




$\{\tfrac{\widetilde{10}}{3}\}$

$t\{\tfrac{\widetilde{10}}{3}\} = \{\tfrac{10}{3}\}$ [smaller in diam.]

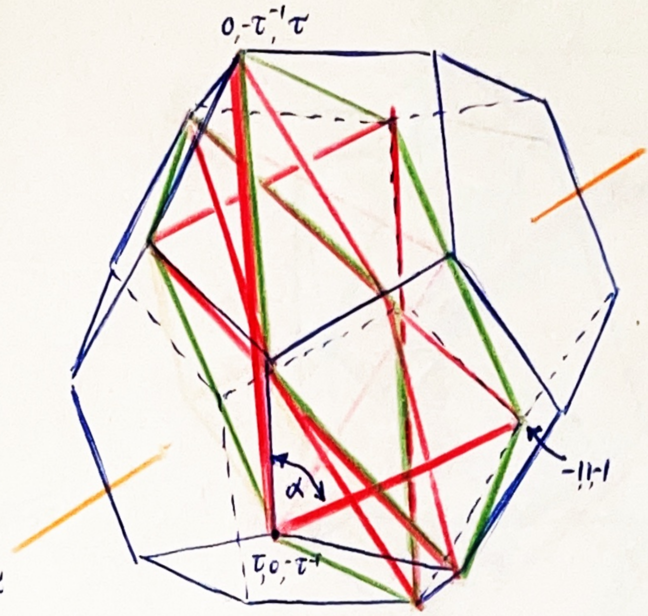
$\{\widetilde{10}\}$



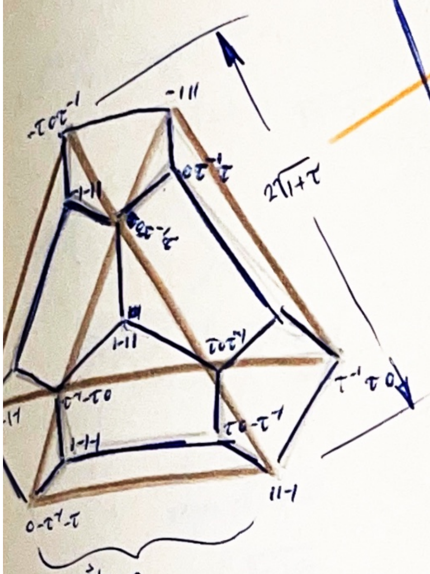
$20\{3\} + 12\{5\}$

$(3\cdot 5)^3$

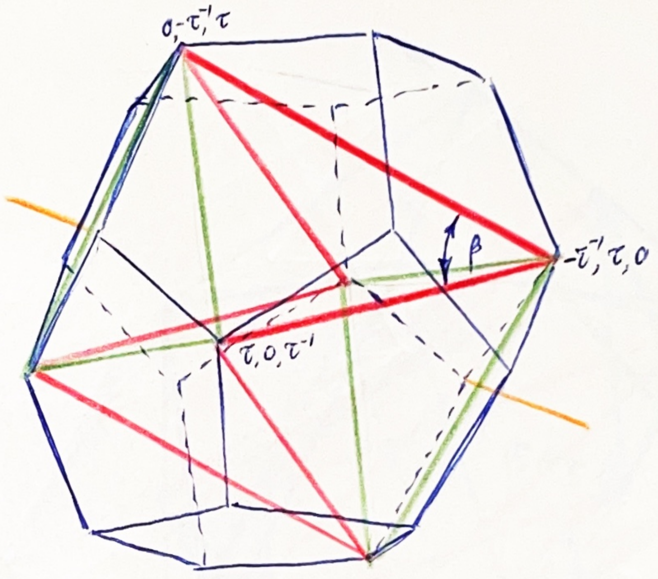
$\checkmark \left\{\tilde{6} \cdot \frac{\tilde{10}}{3}\right\}^3 \qquad 10\{\tilde{6}\} + 6\left\{\frac{\tilde{10}}{3}\right\}$



$\left\{\frac{\tilde{10}}{3}\right\}$



same vertex figure (i.e., geometrically <u>similar</u>) as for $(3\cdot 5)^3$



$\left\{5 \cdot \frac{5}{2}\right\}^3$

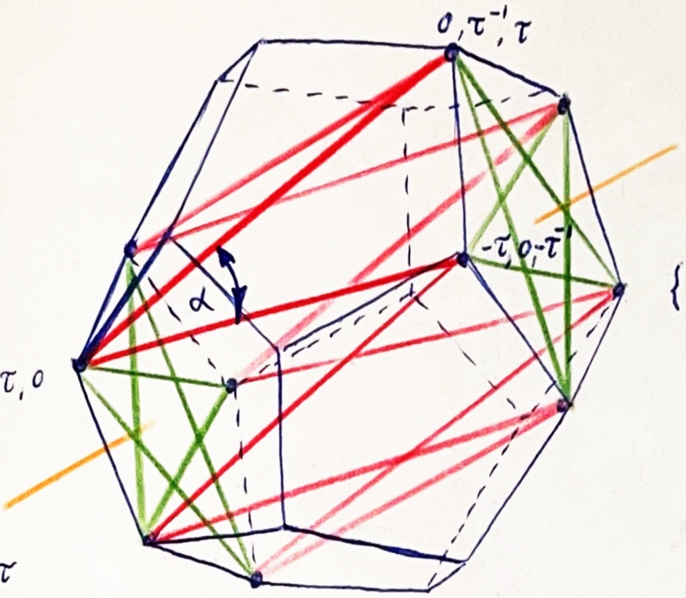
$12\{5\} + 12\left\{\frac{5}{2}\right\}$

$\downarrow \qquad \downarrow$

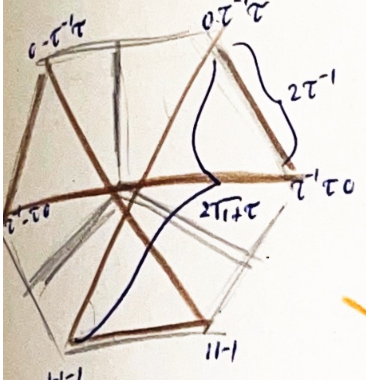
$\left\{\frac{\tilde{10}}{3}\right\} \quad \{\tilde{10}\}$

$\left\{\tilde{10} \cdot \frac{\tilde{10}}{3}\right\}^3$

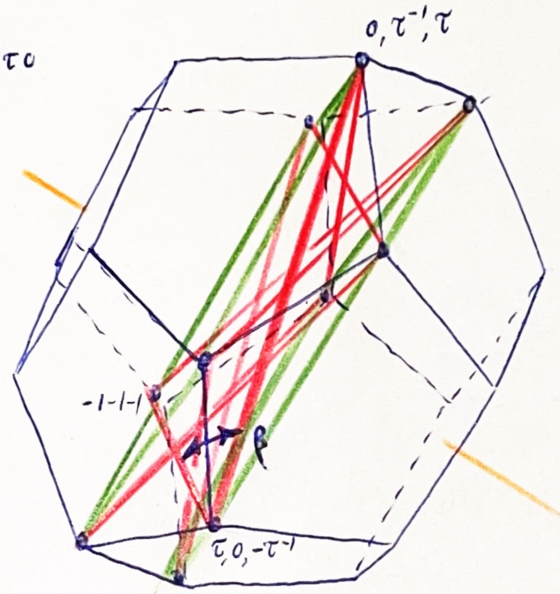
$6\{\tilde{10}\} + 6\left\{\frac{\tilde{10}}{3}\right\}$



$\{\tilde{10}\}$

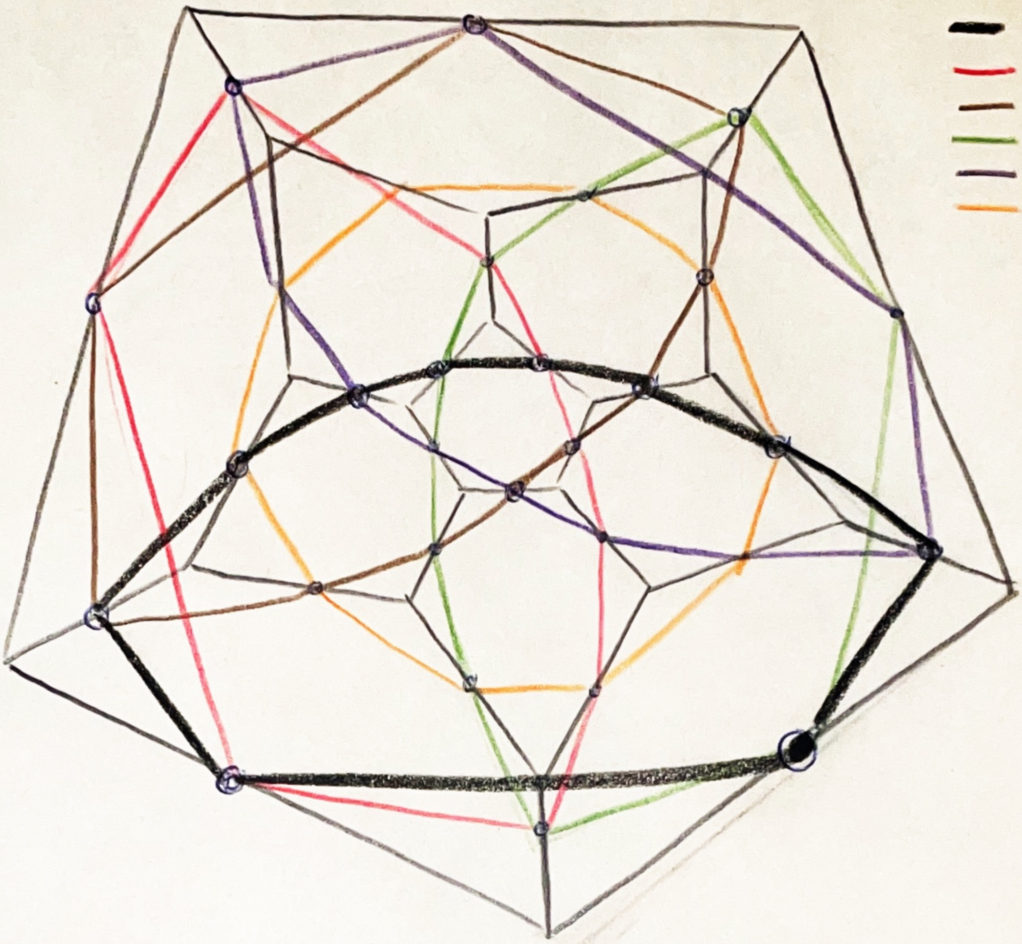


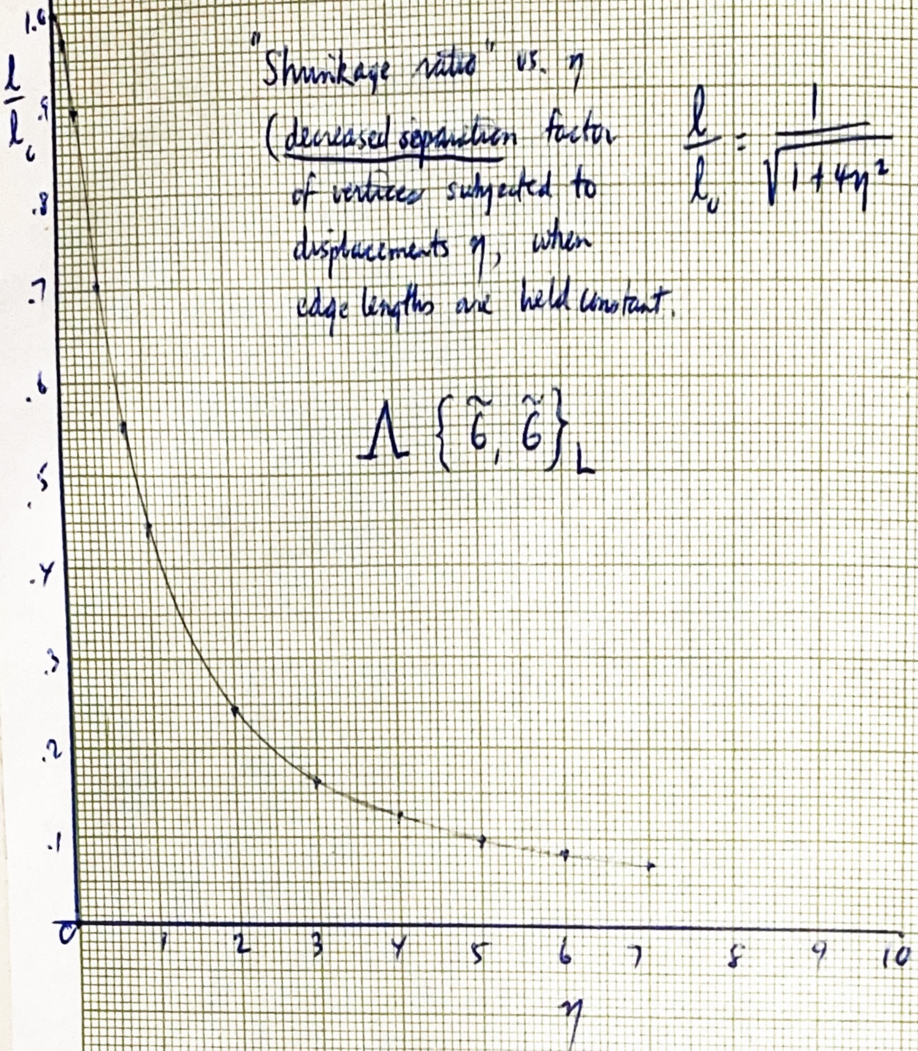
Same (i.e., similar) v.f. as $\left\{5 \cdot \frac{5}{2}\right\}^3$



$\left\{\frac{\tilde{10}}{3}\right\}$

$\frac{3}{2}\,3\,|\,5$ (Fig. 63, Coxeter, Longuet-Higgins, & Miller) $20\{3\} + 6\{10\}$

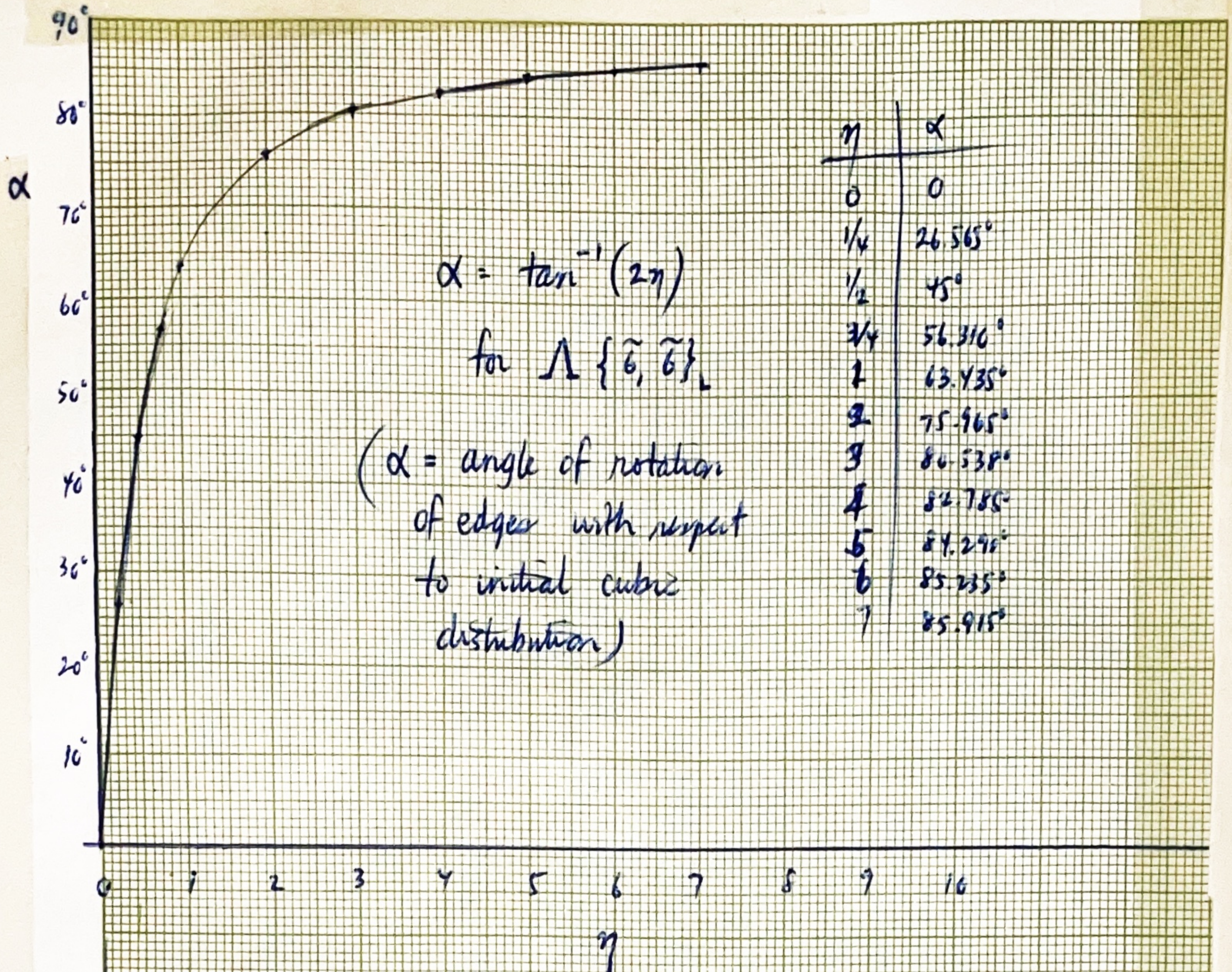




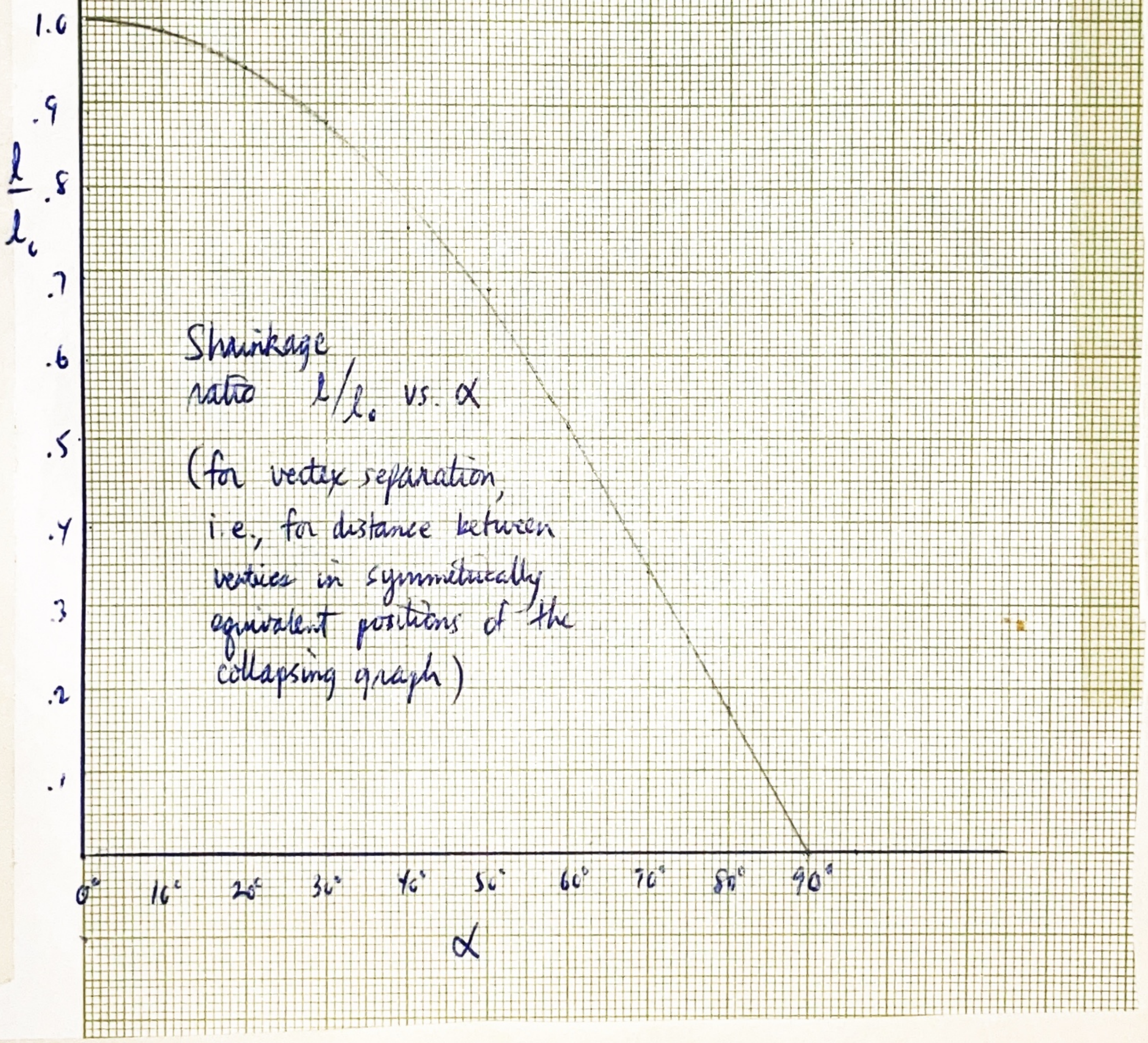
"Shrinkage ratio" vs. η
(decreased separation factor
of vertices subjected to
displacements η, when
edge lengths are held constant.

$$\frac{l}{l_0} = \frac{1}{\sqrt{1+4\eta^2}}$$

$$\Lambda \left\{ \tilde{G}, \tilde{G} \right\}_L$$



η	α
0	0
1/4	26.565°
1/2	45°
3/4	56.310°
1	63.435°
2	75.965°
3	80.538°
4	82.785°
5	84.290°
6	85.235°
7	85.915°



Length of edge

$= \sqrt{30 + 6\eta^2}$

$$\boxed{\Delta\delta = 90^\circ}$$

(starting with the locally centered configuration on the opposite page)

$2 \quad -5 \quad 1$

$-\eta \quad -\eta \quad 0$

$2-\eta, -5-\eta, 1$

$\eta \quad 0 \quad \eta$

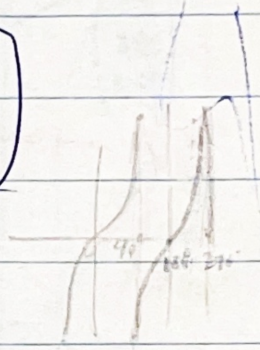
$2-2\eta, -5-\eta, 1-\eta$

$$\cos\delta = \frac{\begin{pmatrix} 2-2\eta \\ -5-\eta \\ 1-\eta \end{pmatrix} \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}}{\sqrt{4(1-2\eta+\eta^2) + (\eta^2+10\eta+25) + (1-2\eta+\eta^2)}}$$

$$= \frac{4-4\eta+25+5\eta+1-\eta}{30\sqrt{1+\frac{\eta^2}{5}}}$$

(Here, η is the amplitude of the rotary skew transformation, starting with the locally centered graph.)

$$\boxed{\cos\delta = \frac{1}{\sqrt{1+\eta^2/5}}}$$

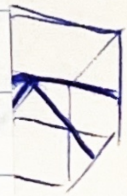


$$\therefore \cos^2\delta = \frac{1}{1+\eta^2/5}$$

$$1 + \frac{\eta^2}{5} = \frac{1}{\cos^2\delta}$$

$$\frac{\eta^2}{5} = \frac{1}{\cos^2\delta} - 1 = \sec^2\delta - 1 = \tan^2\delta$$

When $\eta = 1$, we have reached the "cubic" configuration, with $\sigma_4 = 0$ & $\sigma_6 = \frac{1}{\sqrt{2}}$



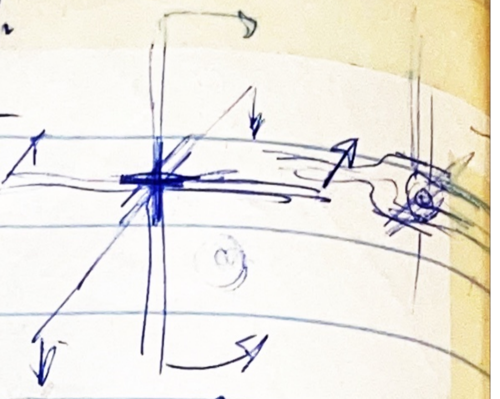
$$\therefore \eta^2 = 5\tan^2\delta$$

$$\left(\cos\varphi_4 = \frac{2(1-\eta)^2}{15+3\eta^2}\right)$$

$$\boxed{\eta = \sqrt{5}\tan\delta}$$

When $\eta = 1$, $\tan\delta = \frac{1}{5}$: $\delta \cong 11^\circ 19'$

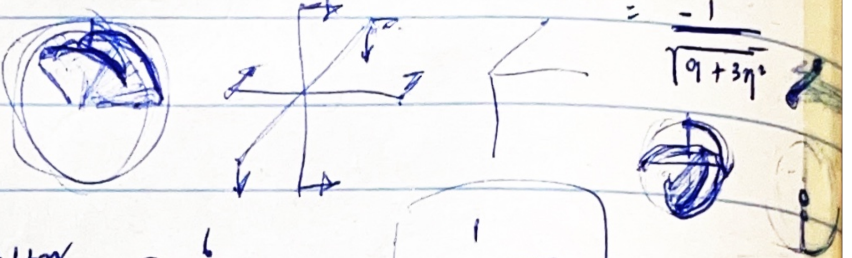
$$\begin{aligned} &1+2\eta+\eta^2 \\ +&1-2\eta+\eta^2 \\ +&4 \\ \hline =&6+2\eta^2 \end{aligned}$$



$$B = \begin{pmatrix} -2 \\ -1+\eta \\ 1+\eta \end{pmatrix}$$

$$\cos\beta = \frac{\begin{pmatrix} -2 \\ -1+\eta \\ 1+\eta \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{6}\cdot\sqrt{6+2\eta^2}} = \frac{+4+1-\eta+1+\eta}{\sqrt{6}\sqrt{6+2\eta^2}} = \frac{-2}{\sqrt{(6)(6+2\eta^2)}} = \cancel{\frac{-2}{2\sqrt{3}(3+\eta^2)}} \; \frac{-2}{2\sqrt{(3)(3+\eta^2)}}$$

$$= \frac{-1}{\sqrt{9+3\eta^2}}$$



$$\cos\beta = \frac{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} -2 \\ -1+\eta \\ 1+\eta \end{pmatrix}}{\sqrt{6}\sqrt{6+2\eta^2}} = \frac{4+1-\eta+1+\eta}{\sqrt{(6)(6+2\eta^2)}} = \frac{6}{\sqrt{(6)(6+2\eta^2)}} = \boxed{\frac{1}{\sqrt{1+\frac{1}{3}\eta^2}}} = \cos\beta$$

$$\begin{vmatrix} i & j & k \\ -2 & -1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = (-2-2, -(-4), -4)$$

$$\cos^2\beta = \frac{1}{1+\frac{1}{3}\eta^2}$$

$$1+\tfrac{1}{3}\eta^2 = \frac{1}{\cos^2\beta}$$

$$\begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 0 & -2 \end{vmatrix} = 4\,4\,4$$

$$3+\eta^2 = \frac{3}{\cos^2\beta}$$

$$\eta^2 = \frac{3}{\cos^2\beta} - 3 = 3\left[\sec^2\beta - 1\right] = 3\left[\tan^2\beta\right]$$

$$\boxed{\eta = \sqrt{3}\tan\beta}$$

$$\beta = \tan^{-1}\frac{\eta}{\sqrt{3}}$$

111 = 0

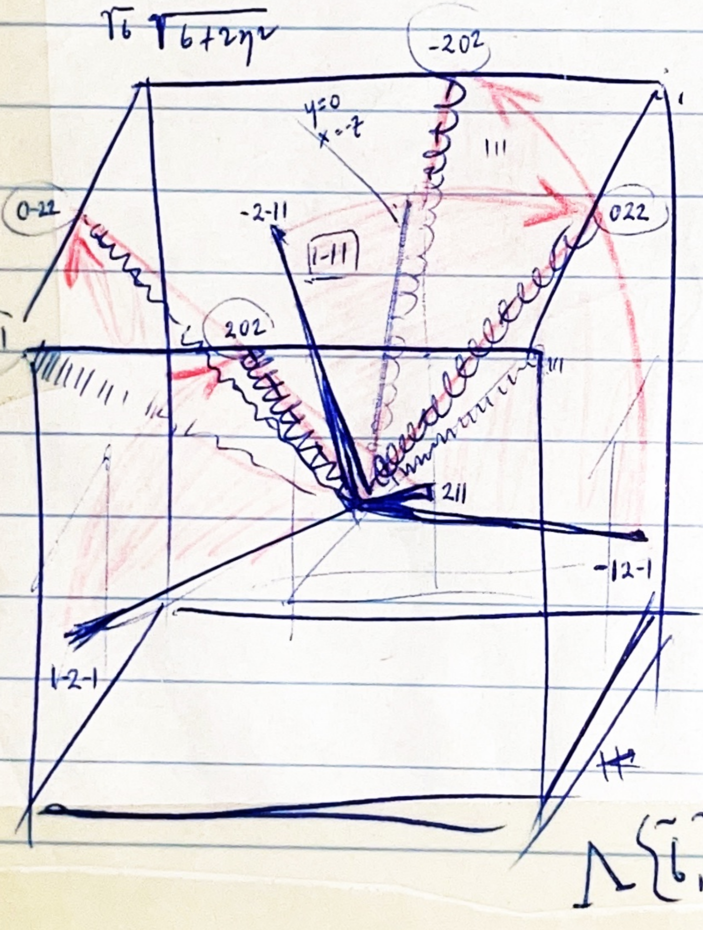
1-11 = 0

2y = 0

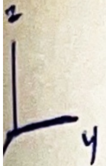
x = -z

$$\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

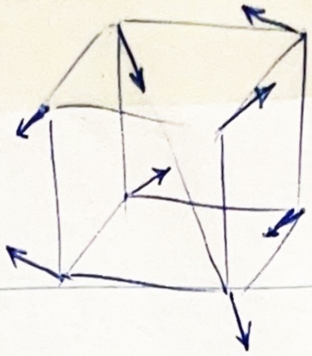
$\Delta\beta = 90^\circ$

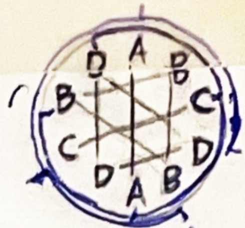


$\Lambda\{[\tilde{i}, \tilde{y}]$

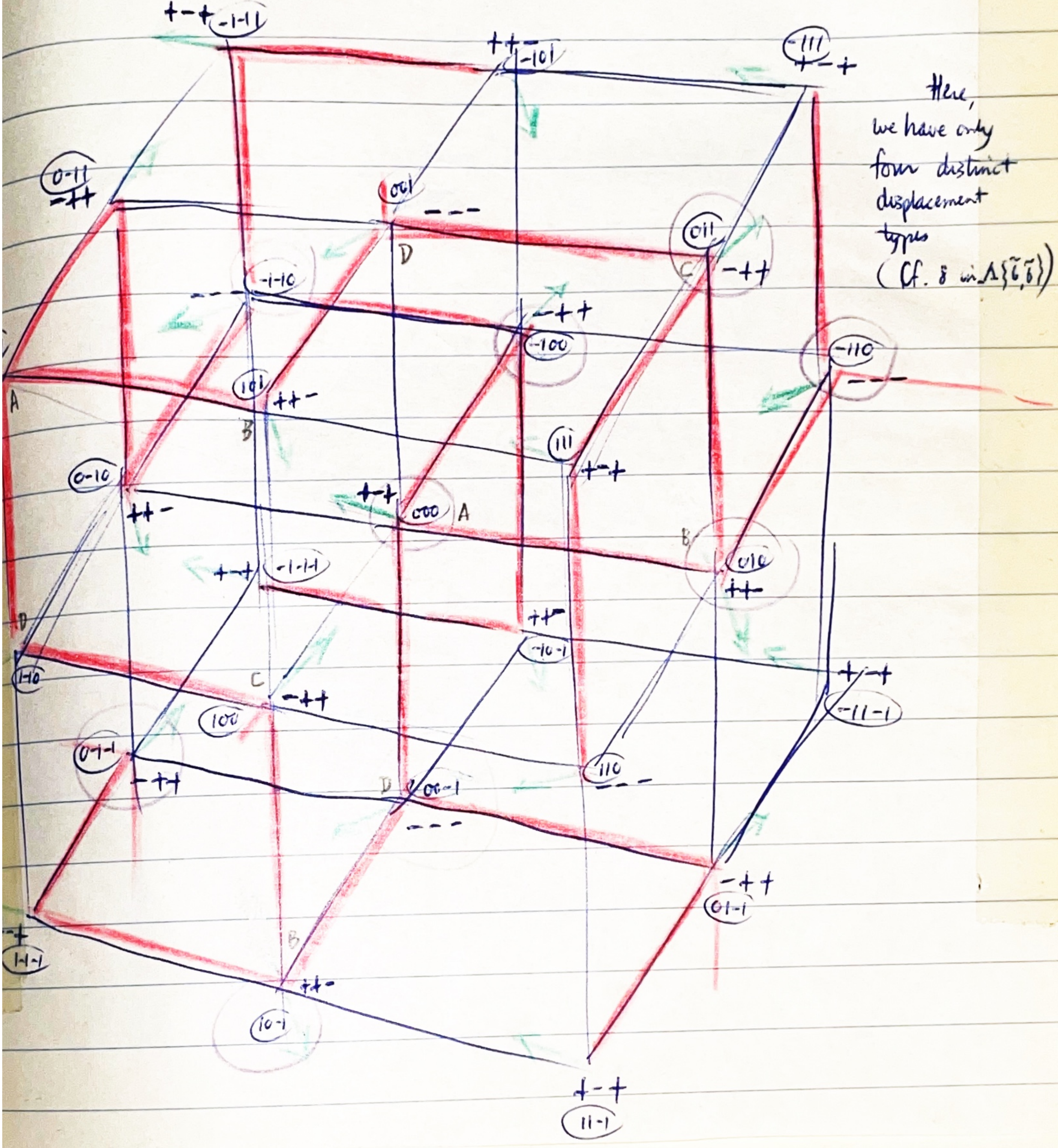


ABCD BADC BDA)





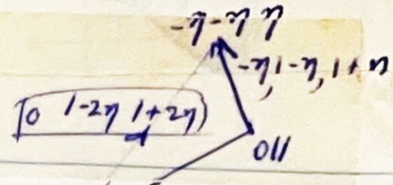
circuit of vertices in 1 decagon

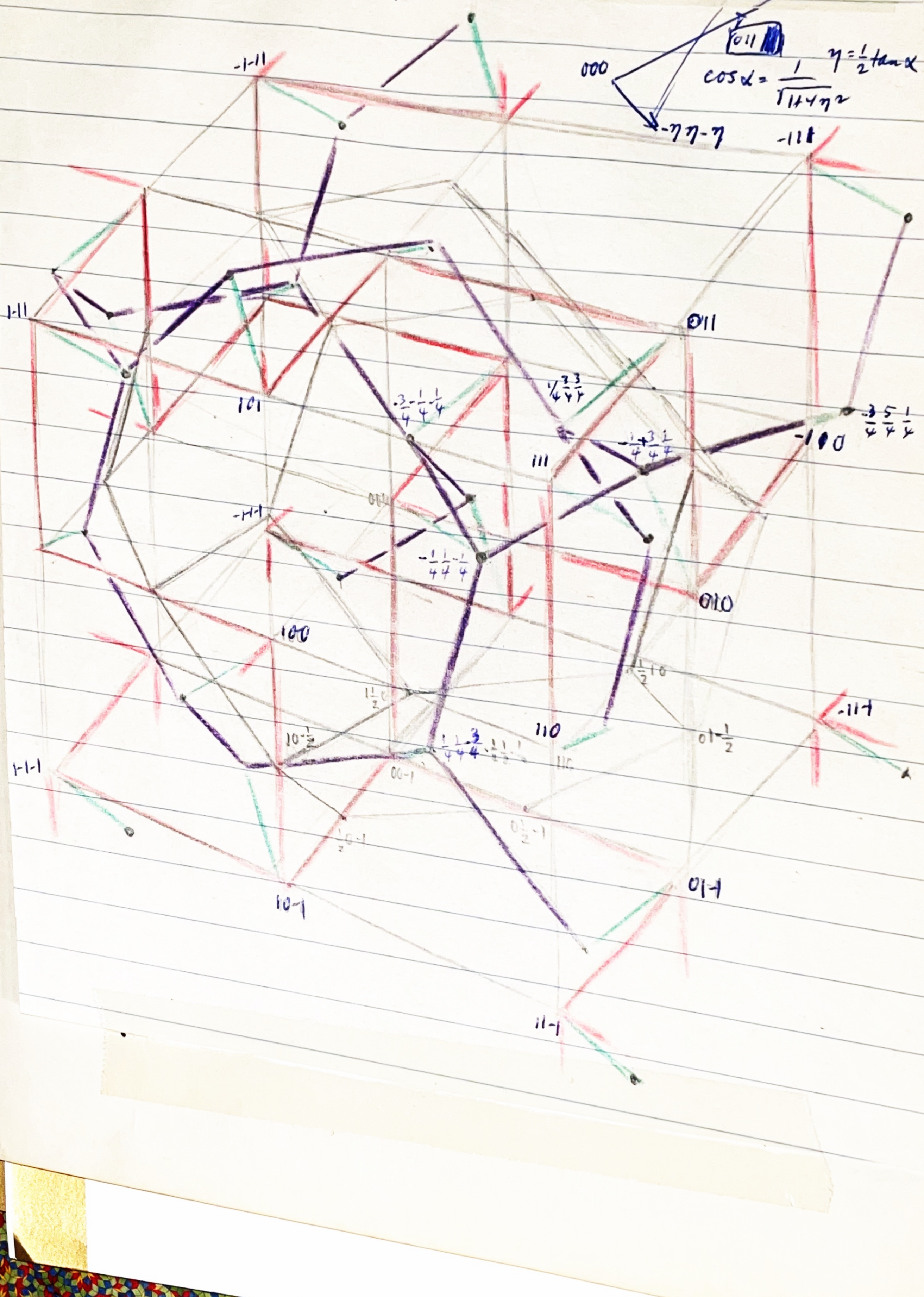


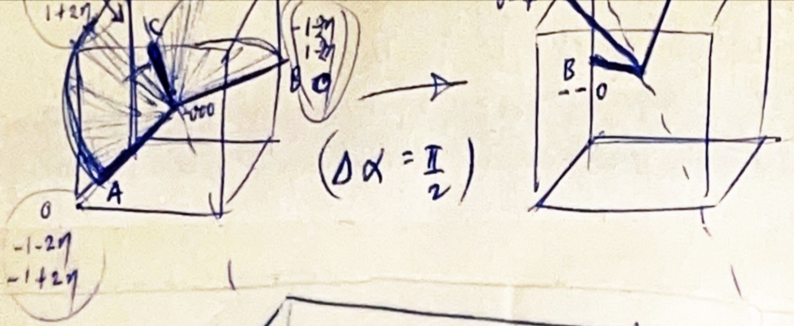
Here, we have only four distinct displacement types (Cf. 8 in $\Lambda\{\tilde{\iota},\tilde{\iota}\}$)

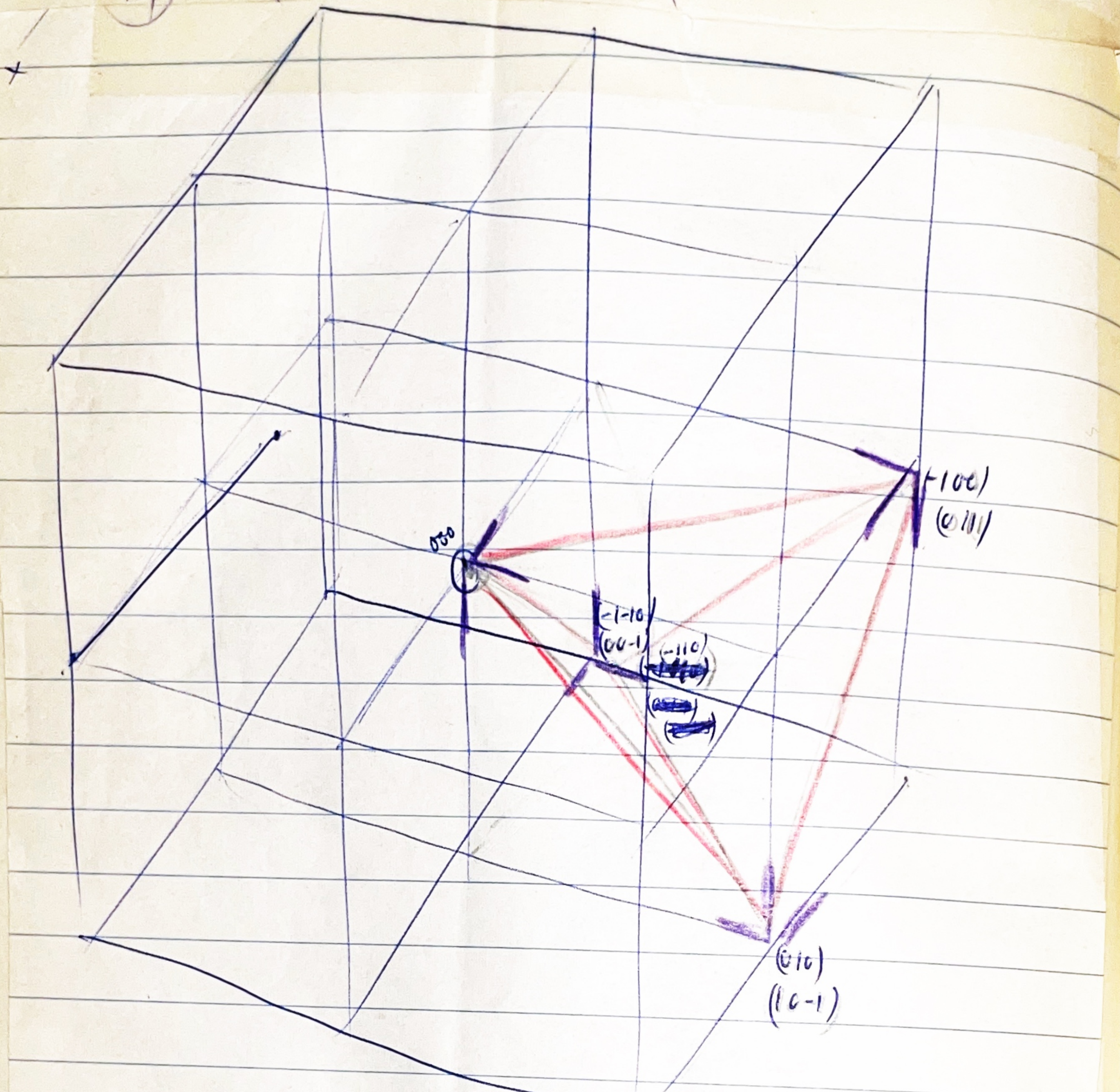
$$\left(\frac{l_0}{l}\right)^{-1} = (\cos\alpha)^{-1} = \sqrt{1+\left(\frac{8}{3}\right)\left(\frac{\lambda}{l_0}\right)^2} \qquad K = \left(\frac{\lambda}{l_0}\,\frac{1}{\tan\alpha}\right) = \frac{\sqrt{6}}{4}$$

Λ (Laves graph) $\rightarrow (SC)_{\text{degree 3}} \rightarrow \{3,3\}$






1+2η
C
B
A
000
0
-1-2η
-1+2η
(Δα = π/2)
B
0


000
(-100)
(011)
(-1-10)
(00-1)
(-110)
(010)
(10-1)

For $\Lambda\,(6\cdot 4)^2_L$, $\cos^2\delta = \dfrac{1-\frac{2}{7}\eta+\frac{1}{49}\eta^2}{1-\frac{2}{7}\eta+\frac{3}{7}\eta^2}$

$1+\tan^2\delta = \sec^2\delta$

$\sec^2\delta - 1 = \tan^2\delta$

$$1-\tfrac{2}{7}\eta+\tfrac{3}{7}\eta^2 = \frac{1}{\cos^2\delta} - \frac{\frac{2}{7}\eta}{\cos^2\delta} + \frac{\frac{1}{49}\eta^2}{\cos^2\delta}$$

$$\cancel{= \sec^2\delta \;\; - \tfrac{2}{7}\eta\sec^2\delta + \tfrac{1}{49}\eta^2\sec^2\delta}$$

$$\therefore\ (\sec^2\delta-1) - \tfrac{2}{7}\eta(\sec^2\delta-1) + \tfrac{1}{49}\eta^2(\sec^2\delta-1) - \tfrac{20}{49}\eta^2$$

$\frac{1}{49}\eta^2(\sec^2\delta) - \frac{21}{49}\eta^2$

$$\tan^2\delta - \tfrac{2}{7}\eta(\tan^2\delta) + \tfrac{1}{49}\eta^2(\tan^2\delta) - \tfrac{20}{49}\eta^2 = 0$$

$$\text{or } \tan^2\delta\left[1-\tfrac{2}{7}\eta+\tfrac{1}{49}\eta^2\right] = \tfrac{20}{49}\eta^2$$

$$\therefore\ \tan^2\delta = \frac{\frac{20}{49}\eta^2}{1-\frac{2}{7}\eta+\frac{1}{49}\eta^2} = \frac{20\eta^2}{49-14\eta+\eta^2} = \frac{(2\sqrt{5})^2\eta^2}{(\eta-7)^2}$$

$-\eta \quad -1+\eta \quad -3$

$\quad c \quad \eta$

η

$-2\eta \quad -1+\eta \quad -3-\eta$

$\quad -1 \quad -3$

2

$-4\eta + 1 - \eta + 9 + 3\eta$

$14 - 2\eta$ ✓

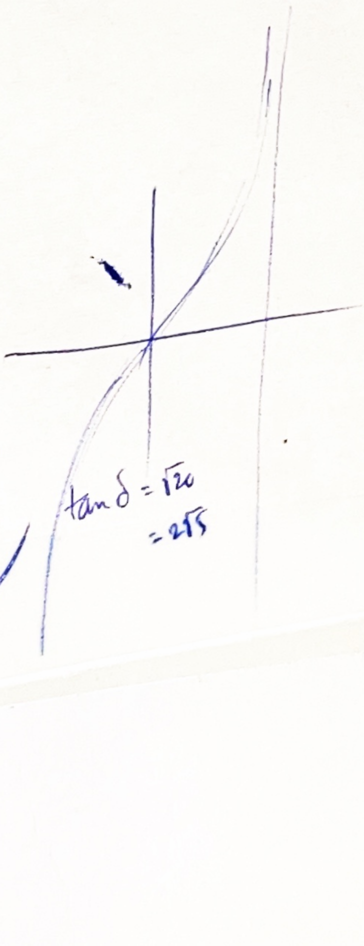
$\eta \to \infty$

$\cos^2\delta = \frac{1}{21}$

$\tan^2\delta = 20$

$21 = \sec^2\delta$

$\frac{1}{21} = \cos^2\delta$ ✓



$\tan\delta = \sqrt{20}$

$= 2\sqrt{5}$

$$\boxed{\tan\delta = \frac{2\sqrt{5}\,\eta}{\eta-7}}$$

$$\boxed{\delta = \tan^{-1}\left(2\sqrt{5}\,\frac{\eta}{\eta-7}\right)}$$ ✓

$(\eta < 0)$

When $\eta \to \infty$,

$\delta = \tan^{-1} 2\sqrt{5}$ ✓

i.e., $\tan^2\delta = 20$

$$\cos\delta = \frac{\begin{pmatrix} 2-2\eta \\ -3-\eta \\ 1-\eta \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}{\sqrt{14}\sqrt{14-4\eta+6\eta^2}}$$

$$4(1-2\eta+\eta^2) + 3(1+2\eta+\eta^2) + 1-2\eta+\eta^2$$

$$= 4-8\eta+4\eta^2 + 3+6\eta+3\eta^2 + 1-2\eta+\eta^2$$

$$= 8-4\eta+8\eta^2$$

$$= \frac{4-4\eta+9+3\eta+1-\eta}{\sqrt{}}$$

$$62 \times 23 = 186 + 124 = 1426$$

$$4(1-2\eta+\eta^2) \quad \cancel{+9+6\eta+\eta}$$

$$(3+\eta)(3+\eta) = 9+6\eta+$$

$$+(9+6\eta+\eta^2) + 1-2\eta+\eta^2$$

$$= 4-8\eta+4\eta^2 + 9+6\eta+\eta^2 + 1-2\eta+\eta^2$$

$$14-4\eta+6\eta^2$$

$$= \frac{14-2\eta}{14\sqrt{1-\frac{2}{7}\eta+\frac{3}{7}\eta^2}}$$

14, 68, 52, 2182

$$= \frac{1-\frac{1}{7}\eta}{\sqrt{1-\frac{2}{7}\eta+\frac{3}{7}\eta^2}}$$

$$\boxed{\frac{1-\frac{\eta}{7}}{\sqrt{1-\frac{2}{7}\eta+\frac{3}{7}\eta^2}} = \cos\delta}$$

$$\cos^2\delta = \frac{1-\frac{2}{7}\eta+\frac{1}{49}\eta^2}{1-\frac{2}{7}\eta+\frac{3}{7}\eta^2}$$

$$\frac{7}{3\cdot 49} \quad \frac{1}{21}$$

When $\eta \to \infty$

$$\cos^2\delta \to \frac{7}{3\cdot 49} = \frac{1}{21} =$$

$$\sigma_6 = \frac{\sqrt{2}(-1+4/3)}{2(1+1/3)} = \frac{\sqrt{2}(1/3)}{2(4/3)} = \frac{\sqrt{2}}{8}$$

$$\cos\delta = .2182$$

$$\boxed{\Delta\delta = 77.3}$$

$$\sigma_4 = \frac{2\sqrt{2}\ \frac{1}{3}}{\sqrt{4(\frac{1}{9})-\frac{4}{3}+2}} = \frac{2\sqrt{2}}{\frac{3}{3}\sqrt{\frac{4}{9}-\frac{12}{9}+\frac{18}{9}}} = \frac{2\sqrt{2}}{\sqrt{10}} = \frac{2\sqrt{20}}{10}$$

When tan δ =

η	$\frac{\eta}{\eta-7}$	$2\sqrt{5}\,\frac{\eta}{\eta-7}$	~~tan~~ δ
0	0	0	0
-1	1/8	$\frac{\sqrt{5}}{4}$ = .559	29.21°
-2	2/9	$\frac{4}{9}\sqrt{5}$ .994	44.83°
-3	3/10	$\frac{6}{10}\sqrt{5}$ 1.3416	53.300°
-4	4/11	$\frac{8}{11}\sqrt{5}$	
-5			
– –	—	$2\sqrt{5}$	77.395°
$-\infty$	1		

$$\varphi = \cos^{-1}\left(\frac{\sigma^2 + \cos\varphi_0}{\sigma^2+1}\right)$$

$\boxed{\text{Calculation of } \{4\} \,\&\, \{6\} \text{ face angles for locally centered configuration of graph}}$

$$\varphi_4 = \cos^{-1}\left[\frac{\frac{26}{13^2} + 0}{\frac{26}{13^2} + 1}\right] = \frac{\frac{2}{13}}{\frac{2}{13} + \frac{13}{13}} = \frac{2}{15}$$

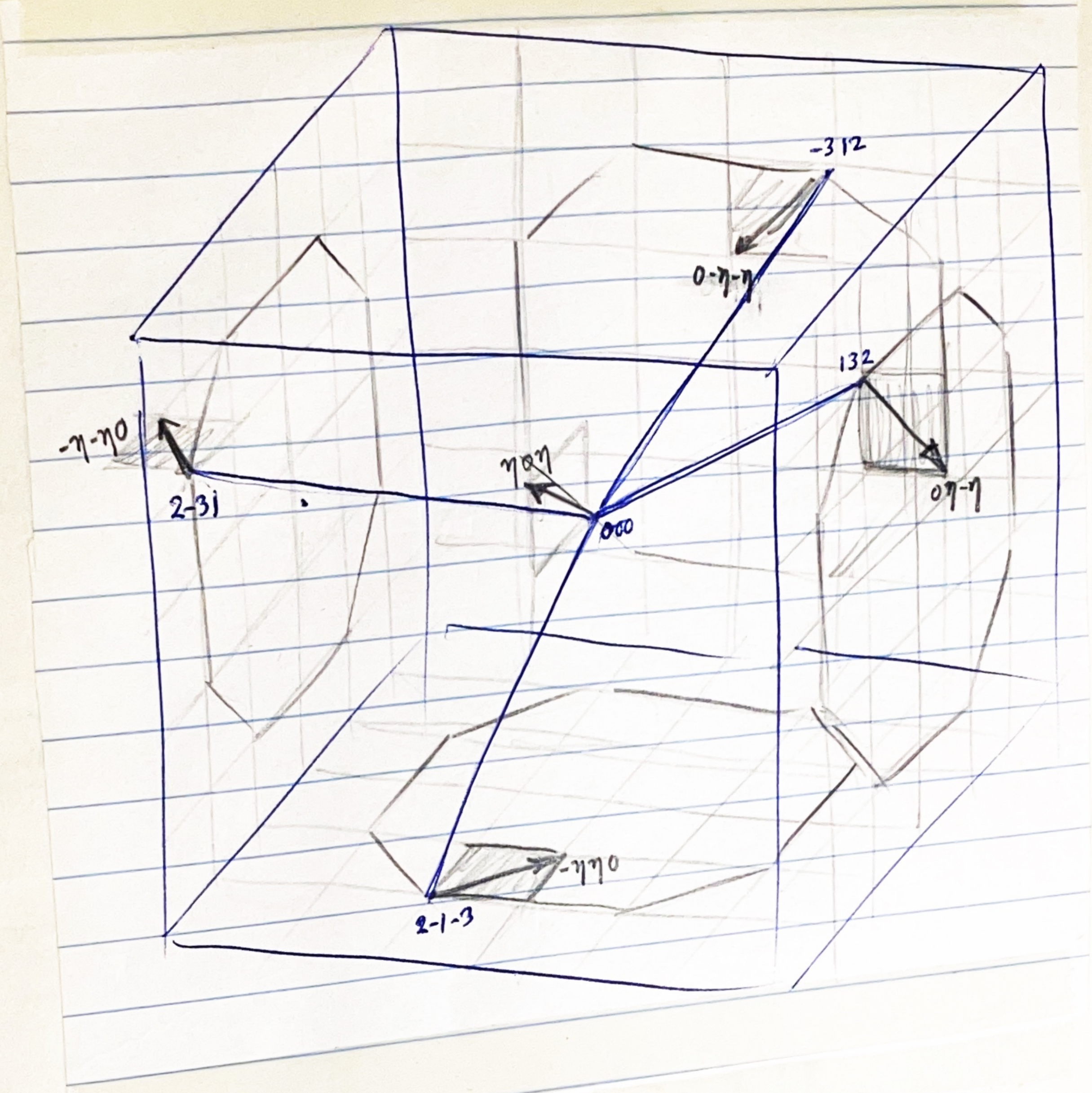
$$\varphi_4 = \cos^{-1}\left(\frac{2}{15}\right) = \cos^{-1}(.1333\ldots) = 82.338^\circ$$

$$\varphi_6 = \cos^{-1}\left(\frac{-13}{30}\right) = \cos^{-1}(-.4333\ldots) = 115.680^\circ$$

$$\varphi_6 = \cos^{-1}\left[\frac{\frac{86}{43^2} - \frac{1}{2}}{\frac{86}{43^2} + 1}\right] = \frac{\frac{2}{43} - \frac{1}{2}}{\frac{2}{43} + \frac{43}{43}} = \frac{\frac{4}{86} - \frac{43}{86}}{\frac{4+86}{86}} = \frac{-39}{90} = -\frac{13}{30}$$

90
64.320

90
25.680
115.680


-312
0-η-η
132
-η-η0
ηοη
2-31
000
0η-η
-ηη0
2-1-3

Arithmetic for shrinkage ratio vs. η for $\Lambda\{\tilde{6},\tilde{6}\}_L$

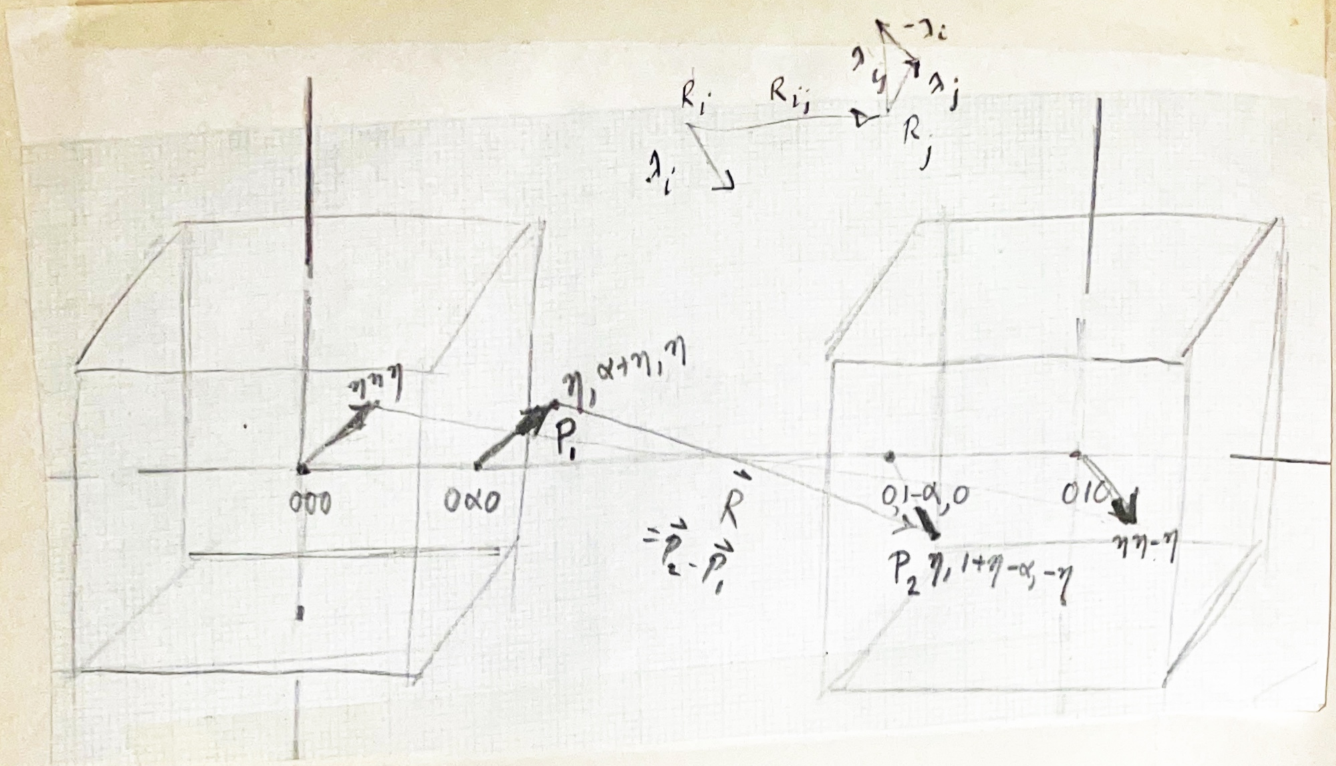
$2\eta = \tan\alpha$

η	2η	α
0	0	0
1/4	1/2	26.565
1/2	1	45°
3/4	1.5	56.310
1	2	63.435°
2	4	75.965
3	6	80.538
4	8	82.785
5	10	84.290
6	12	85.235
7	14	85.915
8	16	
9	18	
10	20	

$\frac{1}{s} =$

$\frac{s}{s_0}$ $\frac{\ell}{\ell_0} = \frac{1}{\sqrt{1+4\eta^2}}$

η	η^2	$4\eta^2$	$1+4\eta^2$	$\sqrt{1+4\eta^2}$	$\frac{1}{\sqrt{1+4\eta^2}}$	
0	0	0	1	1	1	
1/4	1/16	1/4	1.25	1.118	.895	1/8
1/2	1/4	1	2	1.414	.707	
3/4	9/16	2.25	3.25	~~2.245~~ 1.864	.554	
1	1	4	5	2.236	.448	
2	4	16	17	4.123	.243	
3	9	36	37	6.083	.1645	
4	16	64	65	8.062	.1241	
5	25	100	101	10.05	.0994	
6	36	144	145	12.042	.0830	
7	49	196	197	14.036	.0712	



$T_\beta \vec{r}_i = \vec{r}_i \cos\beta + \vec{\Lambda}_i \sin\beta \quad [\delta_i = 0] \qquad \cos\beta = l$

$U_\beta (\vec{r}_i + \vec{\delta}_i) = (\vec{r}_i + \vec{\delta}_i) \cos\beta + \Lambda_i \sin\beta$

$2\eta - 4\alpha\eta$ $\quad 2(\eta - 2\alpha) \quad 4(\eta^2 - 4\eta\alpha + 4\alpha^2)$

$4\eta^2 - 16\eta\alpha + 16\alpha^2$

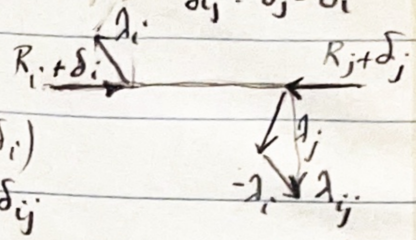
$$\frac{1 - 4\alpha + 4\alpha^2 + 4\eta^2 - 16\eta\alpha + 16\alpha^2}{1 - 4\alpha + 4\alpha^2 + 4\eta^2}$$

$\alpha \to 0: \quad \eta,$

$R_{ij} = R_j - R_i$

$\delta_{ij} = \delta_j - \delta_i$

$$\boxed{(\vec{R}_{ij} + \vec{\delta}_{ij}) \cdot \vec{\lambda}_{ij} = 0}$$



$(R_j + \delta_j) - (R_i + \delta_i) = R_{ij} + \delta_{ij}$

$\lambda_{ij} = \lambda_j - \lambda_i$

ched: $|R| = \sqrt{(1-2\alpha)^2 + (2\eta)^2}$

$\eta - \alpha, -\eta$

$\alpha + \eta, \eta$

$1 - 2\alpha, -2\eta$

$$\boxed{R_0 = (1 - 2\alpha)}$$

~~R_0~~ ~~T_0~~ ~~R'~~

Let $\frac{|R|}{|R_0|} = s =$ stretching ratio

$$= \frac{\sqrt{1 - 4\alpha + 4\alpha^2 + 4\eta^2}}{1 - 2\alpha} = \frac{l}{l_0} = s$$

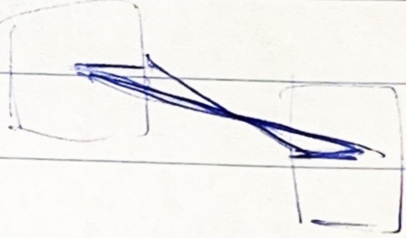
$$\vec{R}' = \frac{\vec{R}}{s} = \frac{(1-2\alpha)}{\sqrt{(1-2\alpha)^2 + (2\eta)^2}} \overbrace{(0, 1-2\alpha, -2\eta)}^{(\vec{P}_2 - \vec{P}_1)}$$

$|R'| = \text{const} = (1 - 2\alpha)$ ✓

motion of R' is in y-z plane (circular arc

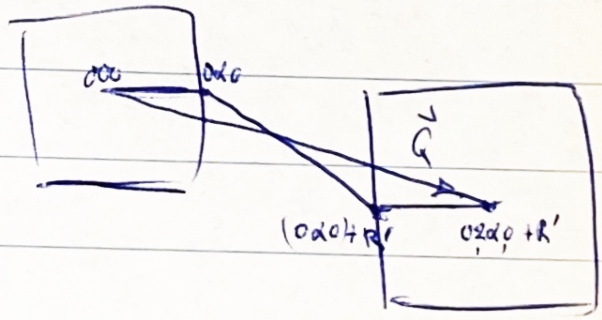
ty center 010 :

$$\begin{array}{ccc} \eta & 1+\eta & -\eta \\ \eta & \eta & \eta \\ \hline 0 & 1 & -2\eta \end{array}$$



ign but. origin & 010 is not const now.

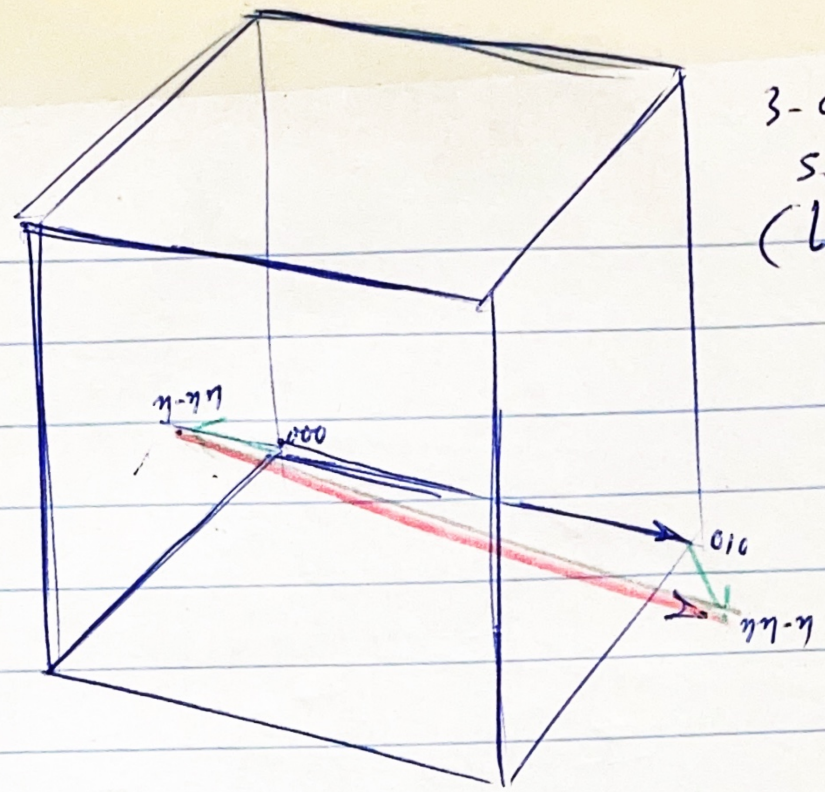
~~Instead, apply~~



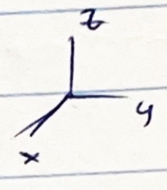
$$\vec{Q} = R' + (0, 2\alpha, 0) =$$

$$\left(\frac{(1-2\alpha)}{\sqrt{(1-2\alpha)^2 + (2\eta)^2}} (0, 1-2\alpha, -2\eta)\right) + (0, 2\alpha, 0)$$

0α0 + R'



3-conn.
s.c.
(Laves)
graph



Place origin of moving coordinate system on η-ηη.

$$\begin{array}{r} \eta,\ 1+\eta,\ -\eta \\ \eta\ \ -\eta\ \ \ \eta \\ \hline 0,\ 1+2\eta,\ -2\eta \end{array}$$

$$l = \sqrt{1+4\eta+4\eta^2+4\eta^2}$$

$$\boxed{\frac{l}{l_0} = \sqrt{1+4\eta+8\eta^2}}$$

$$\boxed{\eta = \frac{1}{2}\left(\frac{\tan\alpha}{1-\tan\alpha}\right)}$$

$$\cos\alpha = \frac{(010)\cdot(0,\ 1+2\eta,\ -2\eta)}{(1)\ \sqrt{1+4\eta+8\eta^2}}$$

$$= \frac{1+2\eta}{\sqrt{1+4\eta+8\eta^2}} = \frac{\cancel{1+2\eta}}{\sqrt{\frac{(1+2\eta)^2+(2\eta)^2}{(1+2\eta)^2}}}$$

$$= \frac{1}{\sqrt{1+\left(\frac{2\eta}{1+2\eta}\right)^2}} = \frac{1}{\sqrt{1+\frac{1}{\left(1+\frac{1}{2\eta}\right)^2}}} = \cos\alpha$$

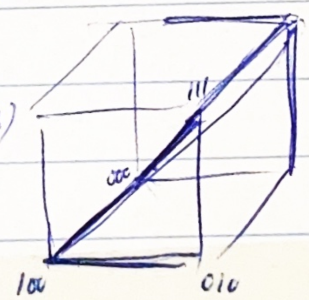
When $\eta = 0$, $\cos\alpha = 1$ $\alpha = 0$

When $\eta = \infty$, $\cos\alpha = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ or $\alpha = 45°$

$$\boxed{\eta = \frac{-\tan^2\alpha + \tan\alpha}{2(\tan^2\alpha - 1)}}$$

i.e., $\eta = \frac{1}{2}\frac{\tan\alpha\ (\cancel{\tan}\ \tan\alpha)}{(1-\tan^2\alpha)}$

$$\boxed{\eta = \frac{1}{2}\left(\frac{\tan\alpha}{1-\tan\alpha}\right)}$$



$\cos\zeta =$

$\frac{(100)(111)}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$= \frac{\sqrt{3}}{3} = .577$

$\frac{.577}{1.732}$

$\therefore \zeta = 35°15'$

Consider 2 vertices separated by 4 cube edge lengths along the y axis, say.

What is their rate of approach?

P_1 & P_2

$\eta \quad 1+\eta \quad -\eta$

$\eta \quad \eta \quad \eta$

$0 \quad 1 \quad -2\eta$

$\sqrt{1+4\eta^2}$

$P_1 = 000 \qquad P_1 \to 000 + \eta\eta\eta = (\eta\eta\eta)$

$P_2 = 040 \qquad P_2 = 040 + \eta\eta\eta = (\eta, 4+\eta, \eta)$

$\vec{P}_2 - \vec{P}_1 = (0, 4, 0)$

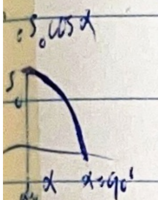
$\boxed{2\eta = \tan\alpha}$

Normalizing factor $= \dfrac{1}{\sqrt{1+4\eta^2}}$

$\therefore$ Separation $= \dfrac{S_0}{\sqrt{1+4\eta^2}} = S_0(1+4\eta^2)^{-1/2} = S_0\left(1 + 4\cdot\frac{1}{4}\tan^2\alpha\right)^{-1/2}$

$= S_0(1+\tan^2\alpha)^{-1/2}$

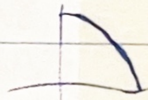
$= S_0(\sec^2\alpha)^{-1/2} = \frac{S_0}{\sec\alpha} = S_0\cos\alpha$



If $\dot{\alpha} = \omega_0 = $ const, & $\eta = \frac{1}{2}\tan\alpha$

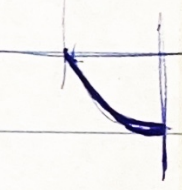
Then $\dot{S} = \cancel{\frac{ds}{d\eta}\frac{d\eta}{d\alpha}} = \dfrac{ds}{d\alpha}\dot{\alpha} = -S_0\sin\alpha\,\omega_0$

$= -S_0\omega_0\sin\alpha$



$\therefore \boxed{\dot{S} = -S_0\omega_0\sin\alpha}$

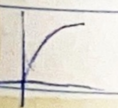
$\ddot{S} = -S_0\omega_0^2\cos\alpha$



What is acceln of 2 points if $\dot{\alpha}$ = const

$\ddot{S} = -S_0\omega_0\cos\alpha\,\dot{\alpha} = -S_0\omega_0^2\cos\alpha$

Starting from collapsed state, $S = S_0\sin\alpha$



$\therefore \dot{S} = S_0\cos\alpha\,\dot{\alpha} = S_0\omega_0\cos\alpha$

$\ddot{S} = -S_0\omega^2\sin\alpha \qquad \therefore \ddot{S}_{\text{initial}} = 0$ Very valuable result

minimal surface theory
could be student of
Bob Osserman

NUMERICAL MATHEMATICS SEMINAR

<u>Graphs, Minimal Surfaces, and Polyhedra</u>

Dr. Alan H. Schoen

National Aeronautics and Space Administration
Electronics Research Center
Cambridge, Massachusetts

Investigation of random and correlated random walks on periodic point arrays in Euclidean 3-space has led to a study of infinite h[illegible]geneous graphs: parallel assemblies of lattices of points joined by equivalent edges. The relationship between such homogeneous graphs, regular polyhedra, and infinite periodic minimal surfaces (IPMS) will be described. It is found that if the faces of a regular polyhedron are not required to be plane, there are 21 examples of regular finite and infinite polyhedra to be added to the conventional list of 15. Two newly discovered IPMS, shown — in collaboration with B. Lawson — to be associate to the two adjoint surfaces of H. A. Schwarz, will be exhibited, together with the Schwarz surfaces. Graph collapse transformations and Dirichlet cell transformations relating infinite and finite polyhedra will be domonstrated by a computer-drawn film. New results on symmetric and random packings of polyhedra will also be described.

TIME: 8:00 P.M., Wednesday, 20 November. 1968

PLACE: Lecture Room, Aiken Computation Laboratory, Harvard University, Cambridge, Massachusetts.

This seminar is sponsored by the local chapters of SICNUM and SIAM. For information contact Donald G. M. Anderson, Aiken Computation Laboratory, Harvard University or Robin E. Esch, Mathematics Department, Boston University.

NOTICE: There will be an informal seminar Friday, 22 November, at 3:00 P.M. in the Conference Room 241 of the Aiken Computation Laboratory. Dr. C. A. Hall of Bettis Atomic Laboratory will discuss "Higher Order Finite Element Methods."

PLEASE CIRCULATE AND POST

Nov, 2068

B. Lawson

PhD student

at Berkeley

specilize the

menemal surface

Theory

coued be student of

Bob Osserman