Wed. noon, Aug. 7, 1968

Dear Lee,

Your letter was handed to me a few minutes before I left the Lab for lunch. It's good to hear from you. Andy (now 6+) came in from Concord to join me for lunch at Harvard Square. Afterward, we're going to the Memorial Church in Harvard Yard to hear an organist practice on the very fine new classical organ there. Meanwhile, with no other paper handy, I'll pen this answer on the back of your letter and Xerox a copy off to you!

Thanks very much for the Escher picture. The Loeb gallery at Hvd. Sq. recently showed ~~an outdoor~~ sidewalk window exhibition of some fascinating original drawings of his — not including this one. Arthur Loeb, no relation to Loeb gallery, is a Dutch emigré and former Harvard and MIT crystallographer who we've become friendly with. He works on stuff a little like mine, but more oriented toward genuine crystallography, at the Kennecott Copper Research Lab in Lexington (nearby). He's a close friend of Escher, and they visited recently in Holland at Escher's home. I have hopes of someday meeting Escher, through Arthur.

The Williams article struck me the same way. Williams is a former student of Peter Pearce's (at San Fern. Valley State College) who got a M.A. in "Design" at So. Ill. Univ. (Fuller's hangout). I found his result interesting, but unconvincing with respect to natural phenomena. Pauling sent Williams a 2-page letter explaining why his results are irrelevant to molecular structure, etc. (non-physically realizable bond angles, for the known elements!).

My work has "exploded" in recent months. I'll enclose 4 abstracts summarizing some of the high points. I hope to visit L.A. in October, to speak at Douglas at the invitation of my old plagiarizing friend Shlichta (up to his old tricks still, but I don't care anymore). If you can arrange it, I'd like to give a 1 hr. seminar at the Physical Research Center. I've finally got some results of apparent utility for the design of expandable space frames. I've also got computer stereo transformation movies of expandable graphs (i.e., space frames or trusses) and of continuous transformations on space-filling polyhedra (very exciting stuff). By October, I'll have data on and pictures of the Dirichlet [polyhedral] cells which are the duals of

Lee:

With less haste — now that I'm no longer without paper in a restaurant at Harvard Square — let me explain that the date of my expected visit to Los Angeles is not yet fixed, but it <u>may</u> be in October. The management of the Douglas Advanced Research Laboratories in Huntington Beach are sponsoring a purely internal company conference on <u>structures</u>, and Schlichta has obtained management approval to invite me to describe my work on nets, polyhedra, and surfaces. Since I work for NASA, I must be careful to avoid violating conflict of interest regulations by disclosing unpublished work of possible commercial value to anybody outside government employment. Since the work I would be describing — both at Douglas and also at TRW, if it seems appropriate for you to arrange something — <u>is</u> already published (it arrived to-day in the Notices [August issue] of the American Mathematical Society, and I will present a 10-min. oral paper on this work on Aug. 28 at Madison), there is no problem here. Nevertheless, I have not yet asked Dr. Van Atta's permission to make the trip. I would prefer not to ask him until I receive a definite formal invitation from Douglas. Thus, I'm merely suggesting that you <u>consider</u> the possibility of arranging to have me give a 1 hr. talk on my work, once it is settled whether I am going to Douglas. I would of course like to visit TRW, and I'm sure a number of people in PRC, plus <u>some</u> of the mechanical design people (especially the physicist, Herb Passen?, in Bldg. M 1 (?) who is a passionate student of unusual topics in geometry — I've forgot his name, but he developed the design for a centrifugally deploying 20-mile-diameter radio telescope antenna for NASA, about 2 yrs. ago at TRW) would find some of my recent work amusing. I now have some reasonable portions of my work in a good state of organization, and it's more intelligible. The parts which are probably of most interest concern continuous, homogeneous transformations of symmetric graphs which leave edge lengths invariant and which describe the total collapse — to a single finite polyhedral cell — of some infinite graphs. 3 of these cases are fully realizable in the form of constructible hardware. The extent of collapse is simply restricted by the finite thickness of the "edges" and "vertices" of the "graph". The joints at the vertices are simple hinge joints — no ball-and-socket swivel is required. These 3 cases, plus 10 other metrically realizable but physically impossible cases (impossible because of periodic collisions of vertices) are all described by the same simple ~~[illegible]~~ equations. In addition, the 3 regular ~~a~~ plane tessellations correspond to graphs which also exhibit the collapse in a physically realizable way. Some mathematicians have told me that these transformations might be of some interest to people who specialize in homology and category theory (whatever they are!), but I doubt it very much. I'll send you a rough informal report I'm writing now, when it's finished.

The other major discovery in this Buckminster Fullerish discipline of mine has been the finding of a very curious infinite periodic minimal surface. The shillelagh for forming the plaster modules will be finished in a few days (after about \$1000.00 worth of machinist labor!). It is a regular skew hexagon with helical "edges" of alternating handedness. Any 2 adjacent edges are 90° helical arcs, of axial length = 2 × radius of the inscribed cylinder, joined with the inscribed cylinders at right angles to one another. I've developed a rigorous proof that there are no other intersection-free infinite periodic minimal surfaces having cubic space lattices besides the 2 Schwarz surfaces (of which you have one specimen) and this new one. Apparently, no straight lines lie on this Laves surface, as I call it. An approximate model I've made, with straight-edged hexagonal modules, is very beautiful. The labyrinths have 3 tunnels coming together at each "open cell" center, instead of 4 and 6, as in the 2 Schwarz surfaces.

There's a good deal more, related to all this work, but it's sketchily described in the 3^d and 4^th of the enclosed abstracts, which I'll probably send to the AMS pretty soon. From now on, I'll be spending almost all of my time — for at least a year or two — writing up all this stuff in full, for reports and math. publications, and looking into the engineering manifestations of the few results of possible practical interest. Next Monday I acquire a full-time associate — 37, Ph.D. math requirements all finished but the thesis, plus an MIT M.S. in mech. engineering (shock phenomena, hyperbolic non-linear differential equations). He'll concentrate on engineering and some further computer graphics work.

I had hoped to find some time here to keep up some bare contact with what people are doing with lasers — especially holograms. However, I've been so exclusively wrapped up in my own work that I've ignored absolutely everything else. It's wonderful to be left so completely alone. The job has been all I hoped for, and more. Van Atta is by far the best boss I've ever had. He's intelligent, tough, witty, friendly, generous, and imaginative. If he has any faults, I haven't discovered them!

Give my best regards to Ralph, Bob Bro., Bob Bri., and Cam. It was wonderful knowing you fellows, and I hope that TRW continues to make it possible for you to work together as long as you care to. Although I was quite unhappy at TRW, that was only because I had almost no interest in the work I was doing (and not very much talent for it either!). I did find it a very stimulating place, loaded with stimulating and likable people. When I stop to think about it, I become very nostalgic. Nevertheless, my satisfaction in being able to spend all my working hours on problems of my own choosing is immense, and it provides great compensation. I have never before had such a long period of uninterrupted satisfaction with my job, and I hope it lasts a while.

You don't have to answer this letter! After the dust settles and I have some definite information about the possibility of a trip to L.A., I'll phone you at work on the gov't. toll line. That will be in several weeks, probably.

Best regards,

Alan

NAME AND TITLE OF AUTHOR

Dr. Alan H. Schoen
Senior Scientist

INSTITUTION OF AUTHOR
(with address and zip number)

NASA-Electronics Research Center
575 Technology Square
Cambridge, Massachusetts

MAILING ADDRESS (if different)

METHOD OF PRESENTATION

By Title

In Person by

TITLE OF PAPER
(Symbols should not be used)

Infinite quasi-regular warped polyhedra (IQRWP) and skewness of regular polygons.

Please check if this abstract is a resubmission or revision of an abstract previously submitted [X]

Fill in below for
IN PERSON ANSTRACTS ONLY

Place of Meeting

Date of Meeting

Meeting Number

* Please notify Secretary in charge of meeting if you later find it impossible to make presentation in person.

Please indicate field of paper

- A [ ] Algebra & Theory of Numbers
- B [ ] Analysis
- C [ ] Applied Mathematics
- D [X] Geometry
- E [ ] Logic and Foundations
- F [ ] Statistics and Probability
- G [ ] Topology
- H [ ] Miscellaneous fields
- I [ ]

TEXT OF ABSTRACT (not more than twenty 65-space lines):

The 9 infinite regular warped polyhedra $\{\tilde{p},\tilde{q}|r\}$[(i) Abstract 658-30, these Notices 107 (1968)] lead to the 15 IQRWP $(\tilde{p}\cdot\tilde{q})$. Of these 15, 9 are truncations $t\{\tilde{p},\tilde{q}|r\}$ or alternations $h\{\tilde{p},\tilde{q}|r\}$[(ii) H.S.M. Coxeter, Regular Polytopes, pp. 145-156]; 11 define continuous spectra of $(\tilde{p}\cdot\tilde{q})$, and result from transformations Λ on $\{\tilde{p},\tilde{q}|r\}$ or on $t\{\tilde{p},\tilde{q}|r\}$.. Λ produces equal translations η of all vertices along the local symmetry axes, adjacent vertices being displaced into alternate "labyrinths". Let the skewness of $\{\tilde{p}\}$ be defined as $\sigma_p \equiv \tan\Theta = \{[\cos\psi + \cos(2\pi/p)]/(1-\cos\psi)\}^{1/2}$, where Θ = the acute angle between an edge of $\{\tilde{p}\}$ and the rotary reflection symmetry plane of $\{\tilde{p}\}$, and ψ=the face angle of $\{\tilde{p}\}$. If every $(\tilde{p}\cdot\tilde{q})$ is identified as $(p[\sigma_p]\cdot q[\sigma_q])^n_X$, where n= the number of $\{\tilde{p}\}$ and of $\{\tilde{q}\}$ at each vertex, and X=P, D, or L [i], then for the 11 "deformable" $(\tilde{p}\cdot\tilde{q})$, σ_p and σ_q are of the form $(a+b\eta)/(c+d\eta+e\eta^2)^{1/2}$ (a,b,c...are constants). If X=L, neither $1/\sigma_p$ nor $1/\sigma_q$ has real zeros; if X=P or D, $1/\sigma_p$ or $1/\sigma_q$ vanishes for some [real] η, corresponding to a "collapsed" $(p\cdot q)$, i.e., a unary space-filling of the interstitial domains of a homogeneous isotropic net [(iii) Abstract 648-106, these Notices 99 (1967), 661].

Schwarz

$F(\alpha,\varphi)$ | $\boxed{P, D \& G}$

$k = \sin\alpha$

$k' = \cos\alpha$

$\Delta(k,\varphi) = \sqrt{1-k^2\sin^2\varphi} > 0$

$K(\tfrac{1}{2}) = F(k=\tfrac{1}{2}, \pi/2)$

NAME AND TITLE OF AUTHOR

A. H. Schoen

Scientist

INSTITUTION OF AUTHOR (with address and zip number)

575 Technology Square
Cambridge, Massachusetts

NASA/Electronics Research Center

MAILING ADDRESS (if different)

575 Technology Square
Cambridge, Massachusetts

METHOD OF PRESENTATION

Title

Person by Alan H. Schoen

TITLE OF PAPER (Symbols should not be used)

Infinite Regular Warped Polyhedra (IRWP) and Infinite Periodic Minimal Surfaces (IPMS)

Please check if this abstract is a resubmission or revision of an abstract previously submitted [X]

Fill in below for IN PERSON ANSTRACTS ONLY

Place of Meeting Madison, Wisconsin

Date of Meeting August 26-30, 1968

Meeting Number

* Please notify Secretary in charge of meeting if you later find it impossible to make presentation in person.

Please indicate field of paper

- A[ ] Algebra & Theory of Numbers
- B[ ] Analysis
- C[ ] Applied Mathematics
- D[X] Geometry
- E[ ] Logic and Foundations
- F[ ] Statistics and Probability
- G[ ] Topology
- H[ ] Miscellaneous fields
- I[ ] ______

TEXT OF ABSTRACT (not more than twenty 65-space lines):

Every IRWP has either skew faces $\{\tilde{p}\}$ or skew vertex figures $\{\tilde{q}\}$ or both; each face is spanned by a minimal surface. If the face and vertex symmetries, R and S [(i) H.S.M.Coxeter, Proc.Lond.Math.Soc. (2), 43 (1937) 33], are rotary reflections, then each face is related to each adjacent face by rotation through a fixed angle about their common edge. RS^{-1} cyclically permutes the edges and vertices of regular r-fold helical polygons $\{r(\omega)\}$ (ω=tangent of the acute angle between any edge of $\{r\}$ and a plane perpendicular to the axis of $\{r\}$), i.e., "holes" [i]. Each $\{p,\tilde{q}|r(\omega)\}$ (regular skew polyhedron [i]): $\{6,\tilde{4}|4(0)\}_P$, $\{4,\tilde{6}|4(0)\}_P$, $\{6,\tilde{6}|3(0)\}_D$; and each $\{\tilde{p},q|r(\omega)\}$ (Schwarz-Neovius IPMS): $\{\tilde{6},4|4(\infty)\}_D$, $\{\tilde{4},6|4(1)\}_D$, $\{\tilde{6},6|3(\tilde{1})\}_P$; has the symmetry of 2 congruent interpenetrating graphs - primitive (P) or diamond (D). A continuous transformation on vertices and edges, which preserves the regularity of $\{\tilde{p}\}$, $\{\tilde{q}\}$, and $\{r\}$, connects the $\{p,\tilde{q}|r\}$ and the $\{\tilde{p},q|r\}$, and reveals the existence of the hitherto unknown $\{\tilde{6},\tilde{4}|4(\sqrt{2}\ )\}_L$, $\{\tilde{4},\tilde{6}|4(\sqrt{2}/2)\}_L$, and $\{\tilde{6},\tilde{6}|3(\sqrt{2}/2)\}_L$ (it also shows that no other intermediate $\{\tilde{p},\tilde{q}|r\}$ exist). These $\{\tilde{p},\tilde{q}|r\}$ correspond to regular maps on an IPMS, L, having the symmetry of 2 interpenetrating enantiomorphous Laves graphs of degree 3[(ii) H.S.M.Coxeter, Can.J.Math. 7, 18-23 (1955)].

(Symbols should not be used)

..Schoen
:ientist
ON OF AUTHOR
s and zip number)

ctronics Research
ter
nology Square
ge, Massachusetts

DDRESS (if different)

OF PRESENTATION

by --

Infinite quasi-regular warped polyhedra (IQRWP) and skewness of regular polygons.

Please check if this abstract is a resubmission or revision of an abstract previously submitted [X]

Fill in below for IN PERSON ANSTRACTS ONLY

Place of Meeting

Date of Meeting

Meeting Number

* Please notify Secretary in charge of meeting if you later find it impossible to make presentation in person.

Please indicate field of paper

A [ ] Algebra & Theory of Numbers

B [ ] Analysis

C [ ] Applied Mathematics

D [X] Geometry

E [ ] Logic and Foundations

F [ ] Statistics and Probability

G [ ] Topology

H [ ] Miscellaneous fields

I [ ]

OF ABSTRACT (not more than twenty 65-space lines):

9 infinite regular warped polyhedra $\{\bar{p},\bar{q}|r\}$[(i) Abstract 658-30, e Notices 107 (1968)] lead to the 15 IQRWP $(\bar{p}\cdot\bar{q})$. Of these 15, re truncations $t\{\bar{p},\bar{q}|r\}$ or alternations $h\{\bar{p},\bar{q}|r\}$[(ii) H.S.M. eter, Regular Polytopes, pp. 145-156]; 11 define continuous spec- of $(\bar{p}\cdot\bar{q})$, and result from transformations Λ on $\{\bar{p},\bar{q}|r\}$ or on $\{\bar{p},\bar{q}|r\}$.. Λ produces equal translations η of all vertices along e local symmetry axes, adjacent vertices being displaced into alrnate "labyrinths". Let the skewness of $\{\bar{p}\}$ be defined as $\sigma_p \equiv \tan\Theta = \{[\cos\psi + \cos(2\pi/p)]/(1-\cos\psi)\}^{1/2}$, where Θ = the acute ngle between an edge of $\{\bar{p}\}$ and the rotary reflection symmetry lane of $\{\bar{p}\}$, and ψ=the face angle of $\{\bar{p}\}$. If every $(\bar{p}\cdot\bar{q})$ is identified as $(p[\sigma_p]\cdot q[\sigma_q])^n_X$, where n= the number of $\{\bar{p}\}$ and of $\{\bar{q}\}$ at each vertex, and X=P, D, or L [i], then for the 11 "deformable" $(\bar{p}\cdot\bar{q})$, σ_p and σ_q are of the form $(a+b\eta)/(c+d\eta+e\eta^2)^{1/2}$ (a,b,c...are constants). If X=L, neither $1/\sigma_p$ nor $1/\sigma_q$ has real zeros; if X=P or D, $1/\sigma_p$ or $1/\sigma_q$ vanishes for some [real] η, corresponding to a "collapsed" $(p\cdot q)$, i.e., a unary space-filling of the interstitial domains of a homogeneous isotropic net [(iii) Abstract 648-106, these Notices 99 (1967),661].

Regular saddle polyhedra (RSP)

Let a RSP be any unbounded surface which has regular skew faces $\{\tilde{p}\}$, each spanned by a minimal surface, with regular plane vertex figures $\{q\}$. There are 12 RSP for which the face and vertex symmetries, R and S [(i) Generators and Relations for Discrete Groups, by H.S.M.Coxeter and W.O.Moser] are rotary reflections and rotations, respectively. To each of the 9 finite regular polyhedra $\{p,q\}_h$ ($\{\tilde{h}\}$ = Petrie polygon of $\{p,q\}_h$ [(i)]), there corresponds a RSP $\{\tilde{h},q\}_p$ with the same edges, vertices, and vertex figures [cf. (i), p. 112]. The faces $\{\tilde{h}\}$ and Petrie polygons $\{p\}$ of $\{\tilde{h},q\}_p$ are the Petrie polygons and faces, respectively, of $\{p,q\}_h$. $\{\tilde{4},3\}_3$, $\{\tilde{6},3\}_4$, $\{\tilde{6},4\}_3$ are limiting transforms ($n\to\infty$) of infinite regular warped polyhedra [(ii) Abstracts 68T-D1 and 658-30, these Notices 107 (1968)]. The 9 quasi-regular polyhedra $t\{\tilde{h},q\}_p = (h\cdot q)$ are identical to the 9 q q|r derived from $(3\cdot3)^2$, $(3\cdot4)^2$, $(3\cdot5)^2$, $(\frac{5}{2}\cdot3)^2$, and $(\frac{5}{2}\cdot5)^2$ by H.S.M.Coxeter et al.[(iii) Phil. Trans. Roy. Soc. Lond. A, 246, 401-450 (1954)]. P.Pearce [(iv) Graham Foundation Research Report (1966)] has described $\{\tilde{4},4\}$ and $\{\tilde{6},3\}$, RSP analogs of regular plane tessellations. The third analog, $\{\frac{\tilde{6}}{2}\cdot6\}$, which was found in collaboration with N.W.Johnson, is an infinite Riemann surface of 2 sheets (cf. H.S.M.Coxeter, Regular Polytopes, p.113).

Quasi-regular saddle polyhedra (QRSP)

Let a QRSP, $(\tilde{p}\cdot\tilde{q})^n$, be any unbounded surface which has regular skew faces $\{\tilde{p}\}$ and $\{\tilde{q}\}$, each spanned by a minimal surface, with cyclic and equiangular plane vertex figures [cf. (i) H. S. M. Coxeter, Regular Polytopes]; n=the number of $\{\tilde{p}\}$ and also of $\{\tilde{q}\}$ at each vertex. Of the 17 known examples of quasi-regular ["flat"] polyhedra $(p'\cdot q')^n$ [(ii) H. S. M. Coxeter et al, Phil. Trans. Roy. Soc. Lond. A, 246, 401-450 (1954)], 9 include equatorial faces (and are truncations of regular saddle polyhedra [(iii) Abstract 68T-D6 these Notices 108 (1968)]). To each of the 8 other $(p'\cdot q')^n$, there corresponds a $(\tilde{p}\cdot\tilde{q})^n$ with a similar vertex figure--$(3\cdot 3)^2$: $(\tilde{6}.\tilde{6})^2$ $[\equiv\{\tilde{6},4\}_3$(iii)]; $(3.4)^2:(\tilde{6}.\tilde{4})^2$; $(3.5)^m:(\tilde{6}.\frac{\tilde{10}}{3})^m$; $(3\cdot\frac{5}{2})^m$: $(\tilde{6}.\tilde{10})^m$; $(5\cdot\frac{5}{2})^m:(\frac{\tilde{10}}{3}\cdot\tilde{10})^m$ (m=2,3). The equatorial polygons (i) of $(\tilde{6},\tilde{4})^2$, $(\tilde{6},\frac{\tilde{10}}{3})^2$, $(6,\tilde{10})^2$, and $(\frac{\tilde{10}}{3},\tilde{10})^2$ are the regular compounds $(2)\{3\}$, $(2)\{\frac{5}{2}\}$, $(2)\{5\}$, and $(2)\ \{3\}$, respectively. Each edge of a $(\tilde{p}\cdot\tilde{q})^n$ joins 2 vertices lying in opposite faces of the corresponding $(p'\cdot q')^n$, and every face of a $\{\tilde{p}\cdot\tilde{q}\}^n$ passes through the center of the $(\tilde{p}\cdot\tilde{q})^n$. $(\tilde{6},\tilde{4})^2$, the only $(\tilde{p}\cdot\tilde{q})^n$ for which N_2 (no. of faces) $\neq N_2$ in the corresponding $(p'\cdot q')^n$, is homeomorphic to $(6\cdot 4)^2$ $(=\frac{4}{3}\,4|3$ [(ii)]$=t\{\tilde{6},4\}_3$ [(iii)]). None of the $(\tilde{p}\cdot\tilde{q})^n$ is orientable.

Issue No. 99

648-106. A. H. SCHOEN, 5126 Elkmont Drive, Palos Verdes Peninsula, California 90274.
ogeneous nets and their fundamental regions.

For the 5 2-dim and 12 3-dim homogeneous nets (connected infinite periodic arrays of nodes, edges joining each node to Z of the Z' nearest nodes [3 ≤ Z ≤ Z'], with all nodes and edges sym- rically equivalent), a fundamental region may be chosen which encloses one node, has Z congruent mirror-symmetric) faces of zero mean curvature, assumes k orientations (k = no. of nodes per t cell), has no edges not symmetry axes of space-filling assembly of such regions (hence symmet- space-linkages, or flexible closed chains of such regions hinged at edges, can be constructed), and s the point symmetry of the net at the enclosed node. This "symmetry domain" is the cell of a mogeneous honeycomb and is constructed as follows: span all edge-circuit polygons of the net, in der of no. of edges, by minimal surfaces, filling the space with closed cells (interstitial domains), ithout reducing volume of already closed cells; join centroids of adjoining cells by line segments; pan smallest polygons of new (reciprocal) net by minimal surfaces, generating congruent cells, or ymmetry domains. In 7 nets, interstitial and symmetry domains are saddle polyhedra (cf. P. Pearce, Synestructics, Graham Foundation, 1966); in 5 3-dim nets, the symmetry domain is the Dirichlet region. (Received June 28, 1967.

648-107. L. S. HUSCH, University of Georgia, Athens, Georgia 30601. Finding a boundary for a 3-manifold.

The following theorem is proved. Let M be a connected, open, orientable 3-manifold with one end. The interior of M is homeomorphic to the interior of a compact 3-manifold if and only if there exists a positive integer N such that every compact subset of M is contained in the interior of a compact 3-manifold M' with connected boundary such that (1) $\pi_1(M - M')$ is finitely generated, (2) genus (bdry M') ≤ N, (3) Every contractible 2-sphere in M - M' bounds a 3-cell. (Received June 19, 1967.)

J. M. GANDHI, York University, Toronto, Ontario, Canada. On Fermat's Last Theorem.

$x^p + y^p = z^p$ where p is a prime > 2 and $z \equiv 0 \pmod p$ (called Case II), x,y,z are ime to each other. We prove the following: If (1) has integral solution in Case II then for following cases (A) for every factor r of x or y, (B) for every factor r of x - y, (C) for r of x + y other than p, it is necessary that $r^{p-1} \equiv 1 \pmod{p^2}$. From these theorems tely follows that if (1) has integral solution in Case II then $2^{p-1} \equiv 1 \pmod{p^2}$ and $1 \pmod{p^2}$. Three different proofs are given for those theorems. Attempts are made to derive following: If (1) has integral solution in Case II then $2^{p-1} \equiv 1 \pmod{p^{p+1}}$, which is impossible us implying the impossibility of Fermat's Last Theorem in Case II. (Received February 11, 1969.)

664-64. ALAN H. SCHOEN, NASA/Electronics Research Center, Cambridge, Massachusetts :139. A fifth intersection-free infinite periodic minimal surface (IPMS) of cubic symmetry. reliminary report.

Of the countably infinite number of IPMS with a cubic Bravais lattice, three with no self-ersections have been known [(i) H. A. Schwarz, Gesammelte Werke, Vol. I, 1890; (ii) E. R. Neovius, iv. of Helsingfors, 1883]. A fourth [(iii) Abstract 658-30, these *Notices* 15 (1968), 727], containing straight lines, is associate to the two [adjoint].Schwarz surfaces. A fifth example, and also the ovius surface, may be obtained as "complements" to the two Schwarz surfaces, respectively, follows: embed, in either Schwarz surface, that pair of dual regular maps [(iii)] whose edges are metry axes of the surface. From the edges of every intersecting pair of Petrie polygons) H. S. M. Coxeter, Regular polytopes] having a common symmetry plane π, construct two gruent skew polygons P and P', separated by π, with alternate vertices at edge midpoints and ices, respectively, of the two Petrie polygons. Span P and P' by minimal surfaces M and M'. complementary IPMS $= \bigcup [M, M']$. The Neovius surface is found to have no associate surface of self-intersections. The new surface probably has neither a locally finite adjoint surface nor intersection-free associate surface. Each Schwarz surface belongs to the same space group as omplement. (Received February 11, 1969.)

664-65. JAMES R. BROWN, Oregon State University, Corvallis, Oregon 97330. Inverse :s, entropy, and weak isomorphism for discrete dynamical systems.

By a discrete dynamical system is meant a quadruple $\Phi = (X, \mathcal{B}, \mu, \varphi)$, where $(X, \mathcal{B}, \mu)$ is a ialized measure space and $\varphi : X \to X$ is a measure-preserving transformation. The entropy e system Φ is denoted $h(\Phi)$. The inverse limit $\Phi = \text{inv lim } (\Phi_n, \psi_n)$ is defined categorically, e Φ_n is a dynamical system and $\psi_n : \Phi_{n+1} \to \Phi_n$ is a measure-preserving transformation for $n = 1,2,\ldots$. The inverse limit is determined uniquely up to isomorphism, and always exists ebesgue systems. We have $h(\Phi) = \lim h(\Phi_n)$, and Φ is (1) ergodic, (2) mixing (of any order), is zero entropy, or (4) has completely positive entropy iff each Φ_n has the same property. cations include systems with quasi-discrete spectrum [L. M. Abramov, Izv. Akad. Nauk SSSR, Mat. 26 (1962), 513-530], quasi-periodic group automorphisms [T. Seethoff, Ph.D. Dissertation, on State Univ., 1968], and the natural extensions of V. A. Rohlin [Izv. Akad. Nauk SSSR, Ser. 25 (1961), 499-530]. A variation of Rohlin's construction is introduced to "lift" a weak isomor-

Schwarz

$F(\alpha, \varphi)\Big|$ $\boxed{P, D, \& G}$

$k = \sin\alpha$

$k' = \cos\alpha$

$\Delta(k, \varphi) = \sqrt{1 - k^2 \sin^2\varphi} > 0$

$$K(\tfrac{1}{2}) = F(k = \tfrac{1}{2}, \tfrac{\pi}{2})$$

$$= F(\underset{[\alpha = 30^\circ]}{\sin\alpha = \tfrac{1}{2}}, \tfrac{\pi}{2})$$

$$= 1.6858$$

$$\boxed{(= 2\,(.842875))}$$

Cf. Schwarz p. 88

$$F(k, \tfrac{\pi}{2}) = K(k)$$

$$K(k') = K'(k)$$

$$K'(\tfrac{1}{2}) = K(\tfrac{\sqrt{3}}{2})$$

$$= F(60^\circ, \tfrac{\pi}{2})$$

$$= 2.1565$$

$$\frac{\omega'}{\omega i} = \frac{K'}{K} = \frac{2.1565}{1.6858} = 1.27926157$$

$$\omega = \frac{1}{2}K \equiv \frac{1}{2} \cancel{K}\, F(k, \tfrac{\pi}{2})\Big|_{k = 1/2} = 0.842875$$

$$i\omega' = i\tau\omega \qquad \boxed{\tau = \frac{iK'}{K}}$$

$$= i\left(\frac{iK'}{K}\right)\omega = -\frac{K'}{K}\omega = -\frac{K'}{K}\left(\frac{1}{2}K\right) = -\frac{1}{2}K' = i\omega'$$

Hence $\boxed{\begin{aligned} \omega' &= \tfrac{1}{2} iK' \\ \omega &= \tfrac{1}{2}K \end{aligned}}$

Schwarz

$F(\alpha, \varphi)|$

$\boxed{P, D, \& G}$

$k = \sin\alpha$

$k' = \cos\alpha$

$\Delta(k, \varphi) = \sqrt{1 - k^2 \sin^2\varphi} > 0$

$$F(k, \tfrac{\pi}{2}) = K(k)$$

$$K(k') = K'(k)$$

$$K(\tfrac{1}{2}) = F(k = \tfrac{1}{2}, \tfrac{\pi}{2})$$

$$= F(\sin\alpha = \tfrac{1}{2}, \tfrac{\pi}{2}) \quad [\alpha = 30°]$$

$$= 1.6858$$

$$\boxed{(= 2\,(.842875))}$$

Cf. Schwarz p. 88

$$K'(\tfrac{1}{2}) = K(\tfrac{\sqrt{3}}{2})$$

$$= F(60°, \tfrac{\pi}{2})$$

$$= 2.1565$$

$$\frac{\omega'}{\omega i} = \frac{K'}{K} = \frac{2.1565}{1.6858} = 1.27926157$$

$$\omega = \frac{1}{2} K \equiv \frac{1}{2} F(k, \tfrac{\pi}{2})\Big|_{k=1/2} = 0.842875$$

$$i\omega' = i\tau\omega \qquad \boxed{\tau = \frac{iK'}{K}}$$

$$= i\left(\frac{iK'}{K}\right)\omega = -\frac{K'}{K}\omega = -\frac{K'}{K}\left(\frac{1}{2}K\right) = -\frac{1}{2}K' = i\omega'$$

Hence
$$\boxed{\begin{aligned} \omega' &= \tfrac{1}{2} i K' \\ \omega &= \tfrac{1}{2} K \end{aligned}}$$

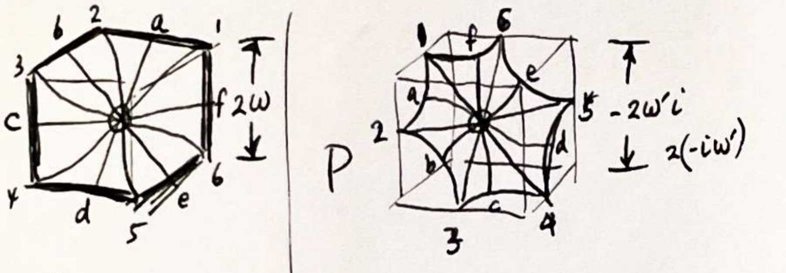
Let $\vec{r}_D = \mathrm{Re}\left\{\begin{array}{l} \int (1-u^2)\, F(u)\, du \\ \int i(1+u^2)\, F(u)\, du \\ \int 2u\, F(u)\, du \end{array}\right\}$, where $F(u) = [1 - 14u^4 + u^8]^{-1/2}$.

$\vec{r}_D(u) \cong \vec{r}_D(v+iw)$ is the parametric representation of the Schwarz surface D.

Then $\vec{r}_P = \mathrm{Re}\,[i\,\vec{r}_D]$ is the parametric representation of P, the surface adjoint to D. The gyroid G, is the associate surface

$$\vec{r}_G = \mathrm{Re}\,[e^{i\theta_G}\,\vec{r}_D], \quad \text{where } \theta_G = \mathrm{ctn}^{-1}(\omega'/\omega i),$$

where 2ω and $2\omega'$ are the fundamental periods of the elliptic functions in terms of which Schwarz also expressed the solutions for P and D.

D 

$\omega \cong 0.842875$

$\omega' = i(1.078258)$

(These are mutually adjoint faces of the "regular maps" $\{6,4|4\}$ on D and P, respectively.)

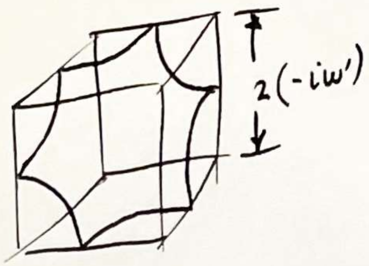
Let $\eta = \omega'/\omega i$ ($\cong 1.27926157$). (Thus, $\theta_G = \mathrm{ctn}^{-1}\eta$.)

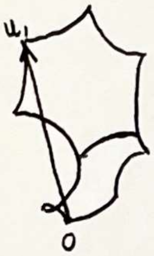
The values of θ for which the corresponding surfaces, associate to P and D, are <u>periodic</u> (i.e., have a <u>discrete</u> ~~sp~~ symmetry group [space group]), are given by

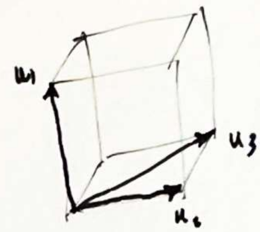
$$\theta = \mathrm{ctn}^{-1}\left[\eta\,\frac{P}{Q}\right],$$ where P and Q have any <u>integer</u> values.

(bounded by a Jordan curve) which corresponds to a self-intersecting minimal surface for all non-zero values of the parameter α (the associate surface transformation parameter in the expression $e^{i\alpha}$), except for $\alpha = \pm(2n+1)\pi$ $(n = 0, 1, 2, \ldots)$.

P





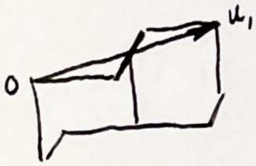


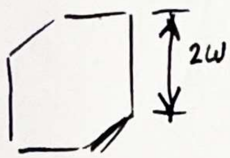
$u_1 = 4(-i\omega')(0\,0\,1)$

$u_2 = 4(-i\omega')(0\,1\,0)$

$u_3 =$ [redacted] $-i\omega')(-1\,1\,0)$

D



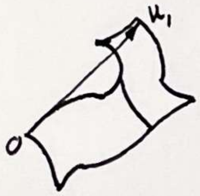


$u_1 = 4$ [redacted] $\omega\,(-1\,1\,0)$

$u_2 = 4$ [redacted] $\omega\,(-1\,0\,-1)$

$u_3 = 4$ [redacted] $\omega\,(-1\,-1\,0)$

[redacted] G



$u_1 = \dfrac{4\omega}{\sqrt{1+\left(\dfrac{\omega}{-i\omega'}\right)^2}}(-$ [redacted] $,$ [redacted] $, 1)$

$u_2 = \dfrac{4\omega}{\sqrt{1+\left(\dfrac{\omega}{-i\omega'}\right)^2}}(-$ [redacted] $,$ [redacted] $, -1)$

$u_3 = \dfrac{4\omega}{\sqrt{1+\left(\dfrac{\omega}{-i\omega'}\right)^2}}(-2, 0, 0)$

Primitive Schwarz surface: $\gamma_p = \dfrac{3\omega}{(-i\omega')} = \dfrac{3(\cancel{\tfrac{1}{2}}K)}{\cancel{(\ )}} = 3\dfrac{i\omega}{\omega'} = \dfrac{K(1/2)}{K'(1/2)}$

~~Diamond~~ $\gamma_{c(p)} = 3\dfrac{K'(\frac{1}{\sqrt{3}})}{K(\frac{1}{\sqrt{3}})}$

$\sqrt{1-(\frac{1}{\sqrt{3}})^2} = \sqrt{1-1/3} = \sqrt{2/3}$

Neovius

If we write: $K'(\frac{1}{\sqrt{3}}) = K(\sqrt{\frac{2}{3}})$

and $K(\frac{1}{\sqrt{3}}) = K'(\sqrt{2/3})$,

then $\gamma_{c(p)} = \dfrac{3\,K(\sqrt{\frac{2}{3}})}{K'(\sqrt{\frac{2}{3}})}$

and $\gamma_p = \dfrac{3\,K(1/2)}{K'(1/2)}$

$\gamma_D = \dfrac{3}{2^{2/3}}\left(\dfrac{\omega'}{i\omega}\right) = \dfrac{3}{2^{2/3}}\dfrac{K'(1/2)}{K(1/2)}$

$\gamma_{c(D)} = ?$

$K = 1.733911$

$\frac{\pi}{2} = 1.5707963$

1.1038420

$\omega' = i\omega\dfrac{K'}{K}$

$-4i\omega'$

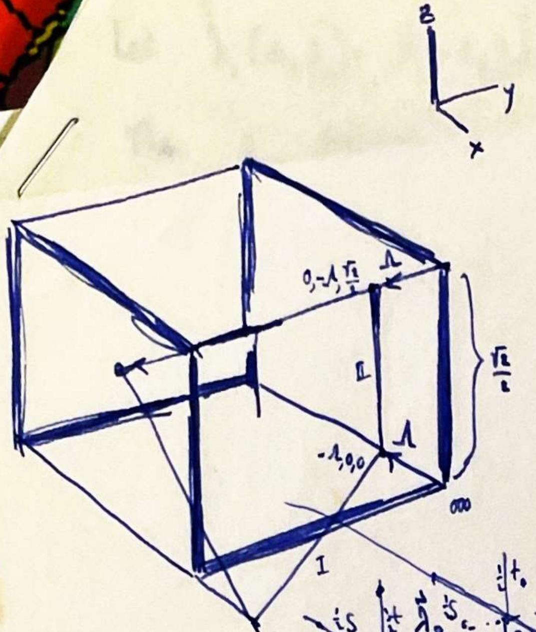
$\omega = \frac{1}{2}K$

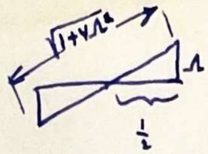
$\omega' = i\left(\frac{1}{2}\cancel{K}\right)\dfrac{K'}{\cancel{K}} = \frac{1}{2}iK'$

$-2\omega' i = -\cancel{2i}\left(\frac{1}{2}iK'\right) = K'$

$$\frac{\left(\frac{S}{V^{2/3}}\right)_{c(p)}}{\left(\frac{S}{V^{2/3}}\right)_{p}} = 1.49694$$

$$\gamma_p = \frac{3\omega}{(-i\omega')} = \frac{3\,\frac{1}{2}K}{\frac{1}{2}K'} = 3\frac{K}{K'} \qquad k = \frac{1}{2}$$





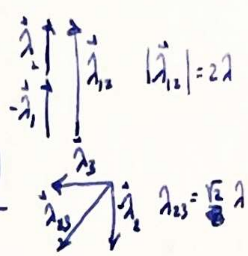
Important

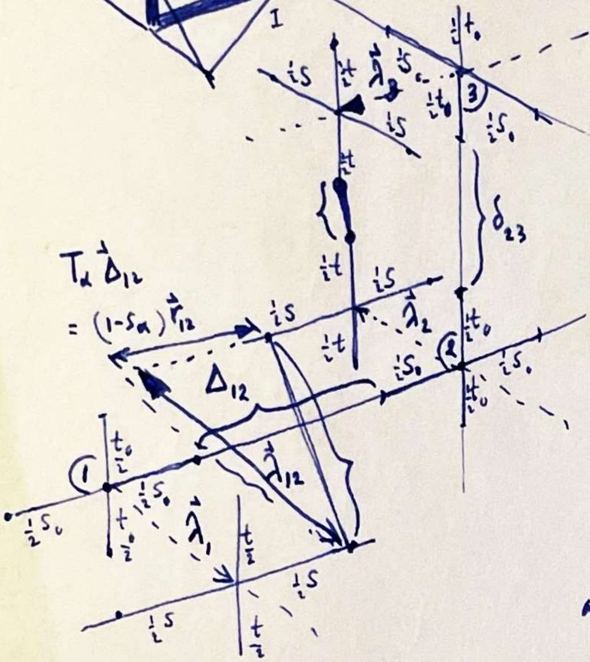
$$\sqrt{\tfrac{1}{4}+\Lambda^2} = \tfrac{1}{2}\sqrt{1+4\Lambda^2}$$

$$|I'| = \boxed{\sqrt{1+4\Lambda^2}}$$

$$|II'| = \sqrt{\Lambda^2+\Lambda^2+\tfrac{1}{2}} = \sqrt{\tfrac{1}{2}+2\Lambda^2} = \boxed{\tfrac{1}{\sqrt{2}}\sqrt{1+4\Lambda^2}}$$

$$\boxed{\frac{I'}{I'_0} = \frac{II'}{II'_0} = \sqrt{1+4\Lambda^2}}$$





Require:

$$\frac{s}{s_0} = \frac{t}{t_0} = \frac{|(1-s_\alpha)\vec r_{12} + \vec\lambda_{12}|}{\Delta_{12} = 1-s_0}$$

$$= \frac{|(1-\frac{t_\alpha}{r_{23}})\vec r_{23} + \vec\lambda_{23}|}{\delta_{23} = l_0 - t_0}$$

(Let l_0 = initial length of vertical edges $|\vec r_{23}|$.)

Now $|(1-s_\alpha)\vec r_{12} + \vec\lambda_{12}| = \sqrt{(1-s_\alpha)^2 + \lambda_{12}^2}$,

since $\vec r_{12}\cdot\vec\lambda_{12} = 0$.

and $|(1-\frac{t_\alpha}{l_0})\vec r_{23} + \vec\lambda_{23}| = \sqrt{(1-\frac{t_\alpha}{l_0})^2 l_0^2 + \lambda_{23}^2}$,

since $\vec r_{23}\cdot\vec\lambda_{23} = 0$.

Then $$\frac{s}{s_0} = \frac{t}{t_0} = \frac{\sqrt{(1-s_\alpha)^2+\lambda_{12}^2}}{1-s_0} = \frac{\sqrt{(1-\frac{t_\alpha}{l_0})^2 l_0^2 + \lambda_{23}^2}}{l_0 - t_0}$$

But $\lambda_{12} = 2\lambda_\alpha$ and $\lambda_{23} = \frac{\sqrt{2}}{2}\lambda_\alpha$

$$\therefore \frac{s_\alpha}{s_0} = \frac{t_\alpha}{t_0} = \frac{\sqrt{(1-s_\alpha)^2 + 4\lambda^2}}{1-s_0} = \frac{\sqrt{(1-\frac{t_\alpha}{l_0})^2 l_0^2 + 2\lambda^2}}{l_0 - t_0}$$

$$\frac{s_\alpha}{s_0} = \frac{\sqrt{(1-s_\alpha)^2+\lambda_{12}^2}}{1-s_0} \qquad \lambda_{23} = \frac{\sqrt{2}}{2}\lambda_{12}$$

$$\frac{t_\alpha}{t_0} = \frac{\sqrt{(1-\frac{t_\alpha}{l_0})^2 l_0^2 + \lambda_{23}^2}}{l_0 - t_0}$$

$$\therefore \frac{\sqrt{(1-\frac{t_\alpha}{l_0})^2 l_0^2 + \frac{1}{2}\lambda_{12}^2}}{l_0 - t_0} = \frac{\sqrt{(1-s_\alpha)^2+\lambda_{12}^2}}{1-s_0}$$

$$\frac{\sqrt{\left(\frac{1-t_\alpha}{t_0}\right)^2 l_0^2 + \frac{1}{2}\frac{\lambda_{12}^2}{t_0^2}}}{\frac{l_0}{t_0}-1} = \frac{\sqrt{\left(\frac{1-s_\alpha}{s_0}\right)^2 + \frac{\lambda_{12}^2}{s_0^2}}}{\frac{1}{s_0}-1}$$

$\therefore \vec{\lambda}_1(\alpha, s_0) = (1-s_\alpha)\vec{\Lambda}_1 \tan\alpha \quad \& \quad \vec{\lambda}_2(\alpha, s_0) = (1-s_\alpha)\vec{\Lambda}_2 \tan\alpha$

Then $\dfrac{s}{s_0} = \dfrac{\sqrt{(1-s_\alpha)^2 + (1-s_\alpha)^2\Lambda_{12}^2 \tan^2\alpha}}{1-s_0} = \dfrac{(1-s_\alpha)}{1-s_0}\sqrt{1+\Lambda_{12}^2\tan^2\alpha}$ But $\Lambda_{12} = 1$

$\Lambda_{12} = 1$

$\therefore \dfrac{s}{s_0} = \dfrac{(1-s_\alpha)}{1-s_0}\sqrt{1+\tan^2\alpha}$

or $\boxed{\dfrac{s}{s_0} = \dfrac{(1-s_\alpha)}{(1-s_0)}\sec\alpha}$

Then we have also $\vec{\lambda}_3(\alpha, t_0) = (1-s_\alpha)\vec{\Lambda}_3 \tan\alpha$ and $\vec{\lambda}_2(\alpha, s_0) = (1-s_\alpha)\vec{\Lambda}_2 \tan\alpha$

$\therefore \dfrac{t}{t_0} = \dfrac{\sqrt{(1-\frac{t_\alpha}{\ell_0})^2\ell_0^2 + (1-s_\alpha)^2\Lambda_{23}^2\tan^2\alpha}}{\ell_0 - t_0}$

But $\Lambda_{23} = \dfrac{\sqrt{2}}{2}$

$\therefore \dfrac{t_\alpha}{t_0} = \dfrac{\sqrt{(1-\frac{t_\alpha}{\ell_0})^2\ell_0^2 + (1-s_\alpha)^2\frac{1}{2}\tan^2\alpha}}{\ell_0 - t_0}$

$= \dfrac{(1-s_\alpha)}{\ell_0 - t_0}\sqrt{\dfrac{(1-t_\alpha)^2}{(1-s_\alpha)^2}\ell_0^2 + \dfrac{1}{2}\tan^2\alpha}$

$= \dfrac{\frac{(1-s_\alpha)}{\sqrt{2}}}{\ell_0 - t_0}\sqrt{2\dfrac{(1-t_\alpha)^2}{(1-s_\alpha)^2}\ell_0^2 + \tan^2\alpha}$

let $\dfrac{2(1-\frac{t_\alpha}{\ell_0})^2}{(1-s_\alpha)^2}\ell_0^2 = 1$

9.43
.707
6501
.501
6.5660 6501
6. 6501
6.5660
6.57

But that implies:

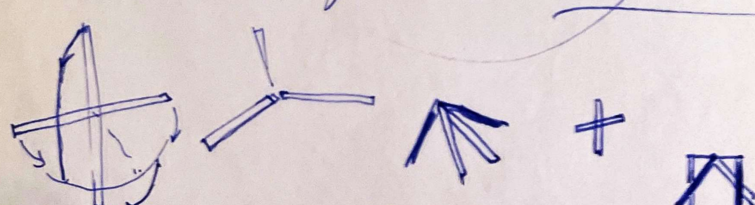
$\dfrac{t_\alpha}{t_0} = \dfrac{(1-t_\alpha)}{\sqrt{2}(\frac{1}{\sqrt{2}} - t_0)}\left[\sqrt{1+\tan^2\alpha} = \sec\alpha\right]$

Then $\ell_0 = \dfrac{1}{\sqrt{2}}\dfrac{(1-s_\alpha)}{(1-t_\alpha)}$ (!)

$\therefore \ell_0 = \dfrac{1}{\sqrt{2}}$ and $s_\alpha = t_\alpha$

or $\dfrac{t_\alpha}{t_0} = \dfrac{(1-t_\alpha)}{1-\sqrt{2}t_0}\sec\alpha$

Cannot satisfy this and $t_\alpha = s_\alpha$ simultaneously !



Let $\vec{\lambda}_1(\alpha, s_0) = (1-s_\alpha)\vec{\Lambda}_1 \tan\alpha$ & $\vec{\lambda}_2(\alpha, s_0) = (1-s_\alpha)\vec{\Lambda}_2 \tan\alpha$

Then $\dfrac{s}{s_0} = \dfrac{\sqrt{(1-s_\alpha)^2 + (1-s_\alpha)^2 \Lambda_{12}^2 \tan^2\alpha}}{1-s_0} = \dfrac{(1-s_\alpha)}{1-s_0}\sqrt{1+\Lambda_{12}^2\tan^2\alpha}$

$\Lambda_{12} = 1$

But $\Lambda_{12} = 1$

$\therefore \dfrac{s}{s_0} = \dfrac{(1-s_\alpha)}{1-s_0}\sqrt{1+\tan^2\alpha}$

or $\boxed{\dfrac{s}{s_0} = \dfrac{(1-s_\alpha)}{(1-s_0)}\sec\alpha}$

Then we have also $\vec{\lambda}_3(\alpha, t_0) = (1-s_\alpha)\vec{\Lambda}_3 \tan\alpha$ and $\vec{\lambda}_2(\alpha, s_0) = (1-s_\alpha)\vec{\Lambda}_2 \tan\alpha$

$\therefore \dfrac{t}{t_0} = \dfrac{\sqrt{(1-\frac{t_\alpha}{l_0})^2 l_0^2 + (1-s_\alpha)^2 \Lambda_{23}^2 \tan^2\alpha}}{l_0 - t_0}$

But $\Lambda_{23} = \dfrac{\sqrt{2}}{2}$

$\therefore \dfrac{t_\alpha}{t_0} = \dfrac{\sqrt{(1-\frac{t_\alpha}{l_0})^2 l_0^2 + (1-s_\alpha)^2 \frac{1}{2}\tan^2\alpha}}{l_0 - t_0}$

$= \dfrac{(1-s_\alpha)}{l_0 - t_0}\sqrt{\dfrac{(1-t_\alpha/l_0)^2}{(1-s_\alpha)^2} l_0^2 + \dfrac{1}{2}\tan^2\alpha}$

$= \dfrac{\frac{(1-s_\alpha)}{\sqrt{2}}}{l_0 - t_0}\sqrt{2\dfrac{(1-t_\alpha/l_0)^2}{(1-s_\alpha)^2} l_0^2 + \tan^2\alpha}$

9.43
707
6501
6501
6.56601 6501
6501
6. 6.5660
6.57

Let $\dfrac{2(1-\frac{t_\alpha}{l_0})^2}{(1-s_\alpha)^2} l_0^2 = 1$

Then $l_0 = \dfrac{1}{\sqrt{2}}\left(\dfrac{1-s_\alpha}{1-t_\alpha}\right)$ (!)

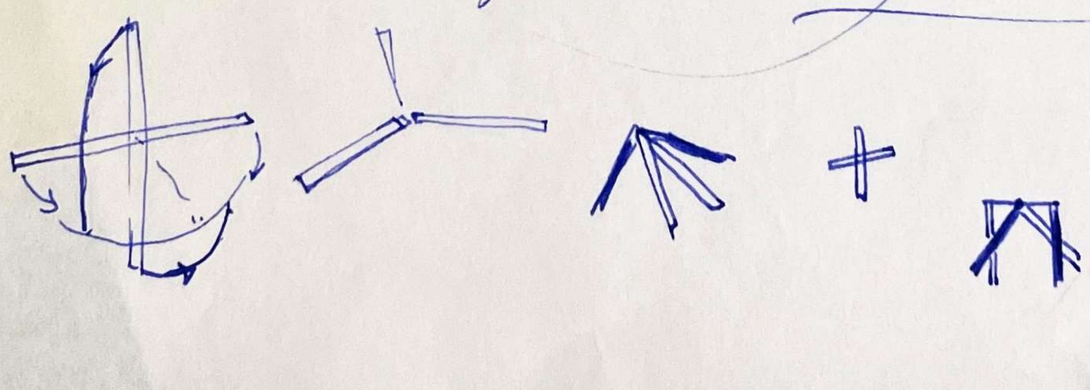
$\therefore l_0 = \dfrac{1}{\sqrt{2}}$ and $s_\alpha = t_\alpha$

But that implies:

$\dfrac{t_\alpha}{t_0} = \dfrac{(1-t_\alpha)}{\sqrt{2}(\frac{1}{\sqrt{2}} - t_0)}\left[\sqrt{1+\tan^2\alpha} = \sec\alpha\right]$

or $\dfrac{t_\alpha}{t_0} = \dfrac{(1-t_\alpha)\sec\alpha}{1-\sqrt{2}\,t_0}$

Cannot satisfy this and $t_\alpha = s_\alpha$ simultaneously!



$$\frac{\left(1-\frac{t_\alpha}{l_0}\right)^2}{(1-s_\alpha)^2}\, l_0^2 = 1 \qquad \text{Suppose } l_0 = \frac{1}{\sqrt{2}} \quad \left(l_0^2 = \frac{1}{2}\right)$$

Then $\dfrac{\left(1-\sqrt{2}\,t_\alpha\right)^2}{(1-s_\alpha)^2} = 1$ If $t_\alpha = \frac{1}{\sqrt{2}} s_\alpha$, then this is satisfied for all α

And $$\frac{t_\alpha}{t_0} = \frac{1-s_\alpha}{\sqrt{2}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} s_0\right]}\sqrt{1+\tan^2\alpha}$$

$$= \underline{\frac{1-s_\alpha}{1-s_0}\sec\alpha}$$