CHECKERBOARD PATTERNS GENERATED BY REFLECTION (a generalization of the ordinary square checkerboard)

It is common knowledge that identical flat tiles of suitable shape can be fitted together to cover a plane surface so as to make a symmetrical pattern. The most symmetrical examples of such tile shapes are the equilateral triangle, the square, and the regular hexagon, but there is no limit to the number of possibilities for the shape of a plane polygon which will fit together with replicas of itself, without overlapping, to cover a plane surface symmetrically. Such a covering of the infinite plane is called a unary plane tessellation.

We are interested in a ~~particular kind~~ restricted type of unary plane tessellation, which we will call a checkerboard generated by reflection. We define such a pattern by the following rules:

1) ~~Each separate~~ every tile is an exact replica of a tile, which is colored black on one s[illegible] and white on the other, and whose boundary [illegible] straight-edged plane polygon, not necessarily c[illegible]

2) every tile in the tessellation is related by reflection in each of its boundary edges to an adjacent tile (two tiles are defined as adjacent if they ~~[illegible]~~ have at least one boundary edge in common);

3) no finite region of the tessellation is covered by more than one tile, i.e., no portions of different tiles overlap except for boundary edges and boundary points;

4) the color of every tile in the tessellation is "opposite" to that of each adjacent tile (as in a square checkerboard, for example).

Reflection in a line, in the sense of rule 2, is illustrated in Fig. 1. The two polygons α and β of Fig. 1 are related by reflection in the boundary edge AB. Any two corresponding points of the two polygons, e.g., $P(\alpha)$ and $P(\beta)$, are equidistant from the extended line through AB and lie on a line perpendicular to that extended line.

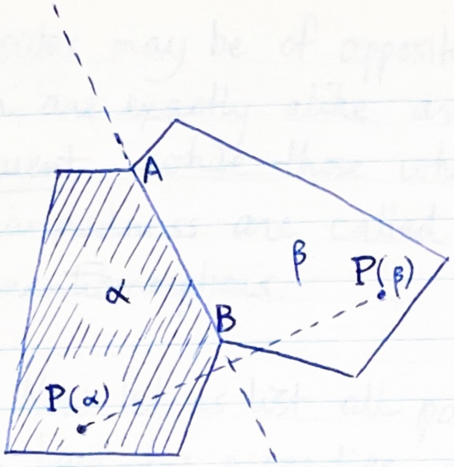


Fig. 1
Two polygons — α and β — which are related by reflection in their common boundary edge AB

We could omit rule 4 if we extended the definition of reflection in rule 2 to include color reversal. Alternatively, we could replace both rules 2 and 4 by a single rule which says:

every tile in the tessellation is related by rotation, through a half-turn (i.e., 180°) about each of its boundary edges, to an adjacent tile.

It is a matter of taste which mode of description we choose to adopt. The results are the same in either case.

It is apparent that this kind of checkerboard generated by reflection looks the same on both sides. However, we must admit the possibility that the patterns on the two sides may be of opposite handedness. Patterns which are exactly alike are called directly congruent, while those which are alike except for handedness are called oppositely congruent, or enantiomorphous.

Now let us list all possible examples of tiles which have the properties defined by rules 1 to 4. First we will consider only the kind of plane polygon whose boundary is a simple closed curve, i.e., a curve which does not touch itself at any point. We find that the only examples of such elementary tiles are the following convex polygons:

- rectangle;
- square;
- equilateral triangle ($60^\circ - 60^\circ - 60^\circ$);
- isosceles right triangle ($45^\circ - 45^\circ - 90^\circ$);
- $30^\circ - 60^\circ - 90^\circ$ triangle.

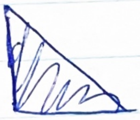
These elementary tiles are shown in Fig. 2. Fig. 3 shows the corresponding checkerboards.

Note that the regular hexagon does not satisfy the rules for a checkerboard generated by reflection. The elementary tiles which qualify are those whose face angles are equal

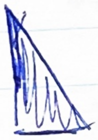
(a) rectangle : R

(b) square : S

(c) equilateral triangle : T_{333} ~~(60°-60°-60°)~~



(d) isosceles right triangle : T_{244} ~~(45°-45°-90°)~~

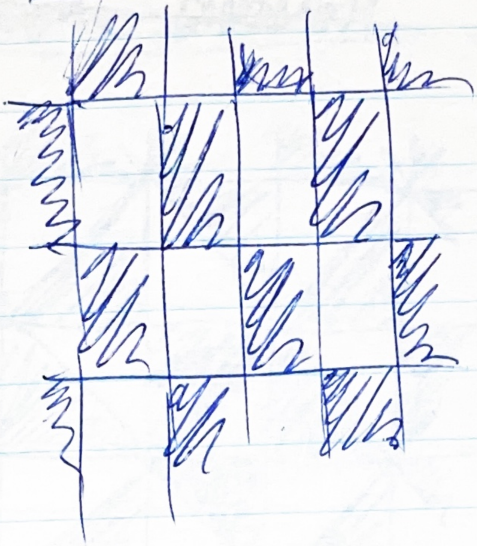


(e) " 30°-60°-90° triangle : T_{236}

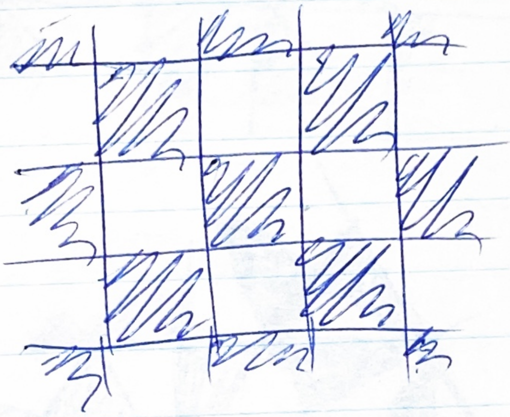
Fig. 2 Elementary tiles of checkerboards generated by reflection.

Checkerboards composed of elementary tiles

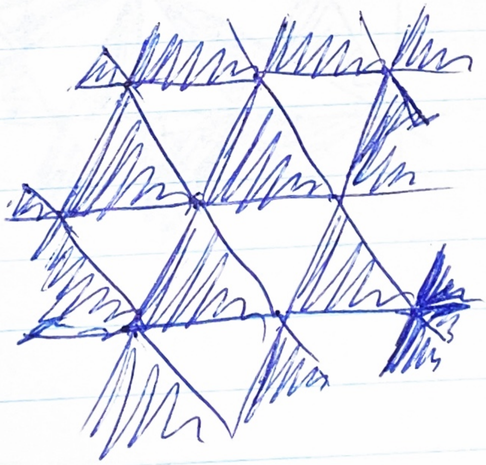
Fig. 3



(a) R



(b) S

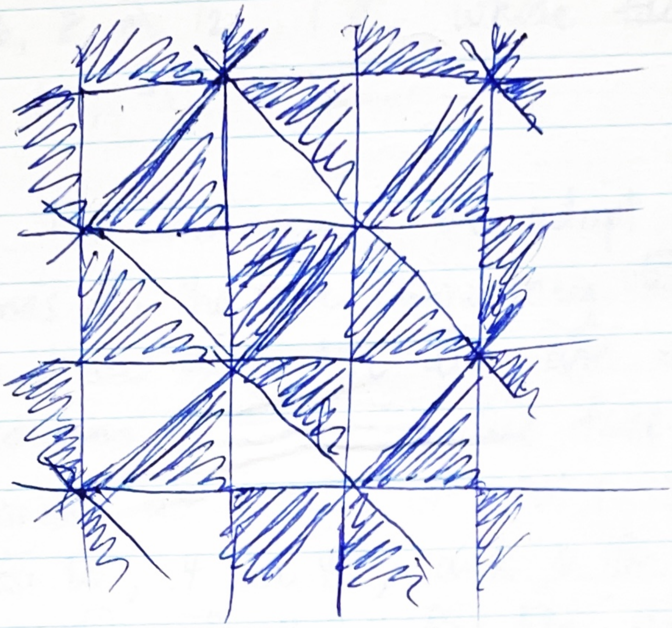


(c) T_{333}

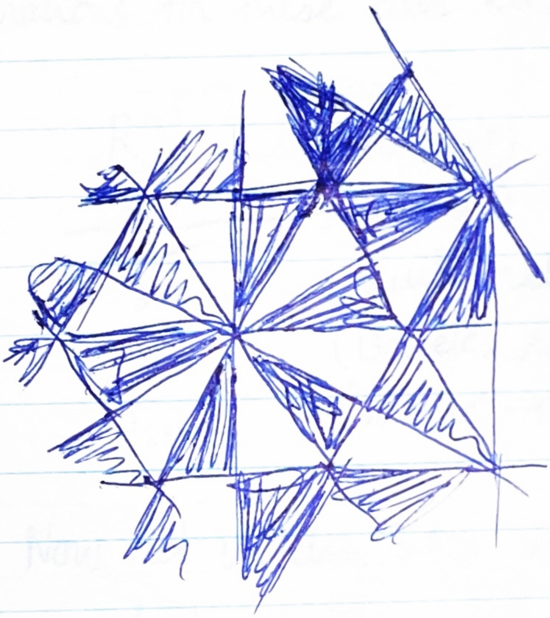
Fig. 3 (concl.)

~~HISTORICAL SUMMARY~~

(d) T_{244}



(e) T_{236}



to 360° divided by one of the even numbers 4, 6, 8, or 12, i.e., whose face angles are 90°, 60°, 45°, or 30°.

For convenience, we adopt abbreviated names for the five elementary tiles of Fig. 2. The three triangular tiles are numbered according to their respective face angles, expressed as factors of 180°: 2 for 90°, 3 for 60°, 4 for 45°, and 6 for 30°. The abbreviations for these five tiles are:

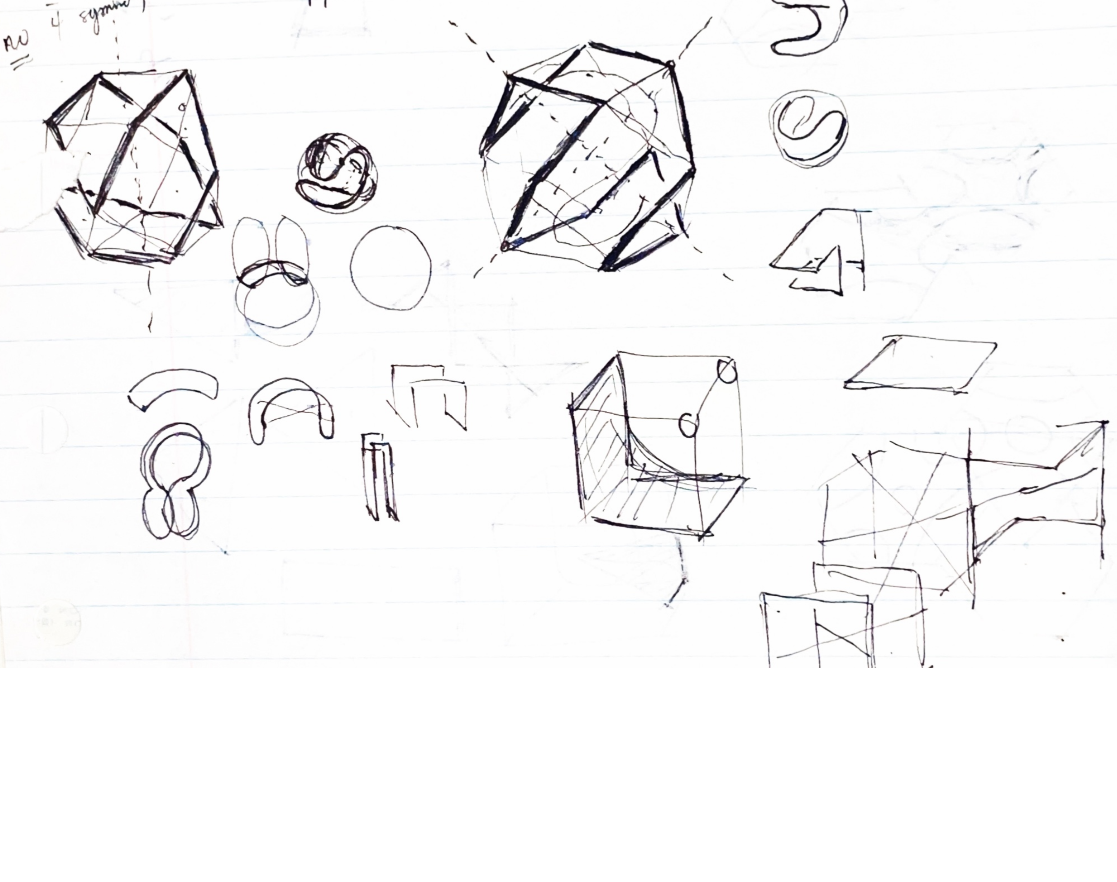
R	(rectangle)
S	(square)
T_{333}	(equilateral triangle)
T_{244}	(isosceles right triangle)
T_{236}	(30° - 60° - 90° triangle).

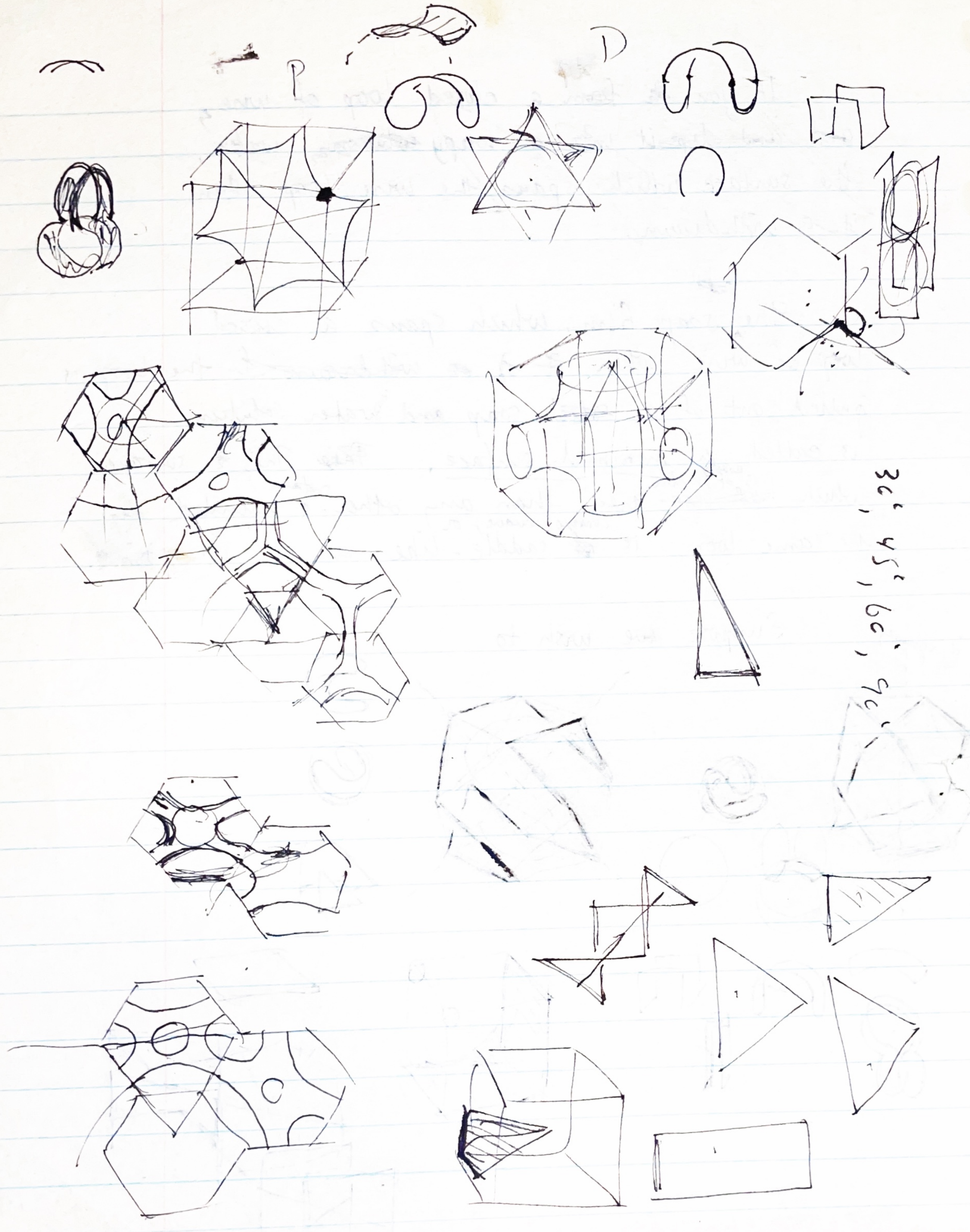
Now let us drop the restriction to simple polygons, which we have called elementary tiles. If we admit polygons whose boundaries touch themselves at some points, we find eight additional tiles. We will call these compound tiles, because they consist of aggregates of the elementary tiles we have already considered. In each case, the boundary is a polygon which touches itself at only one point.

If you ~~di~~ ~~form~~ dip a closed loop of wire, ~~and~~ ~~and dip~~ it into ~~a~~ soapy ~~solution~~ or water, the surface which spans the wire loop when it is withdrawn

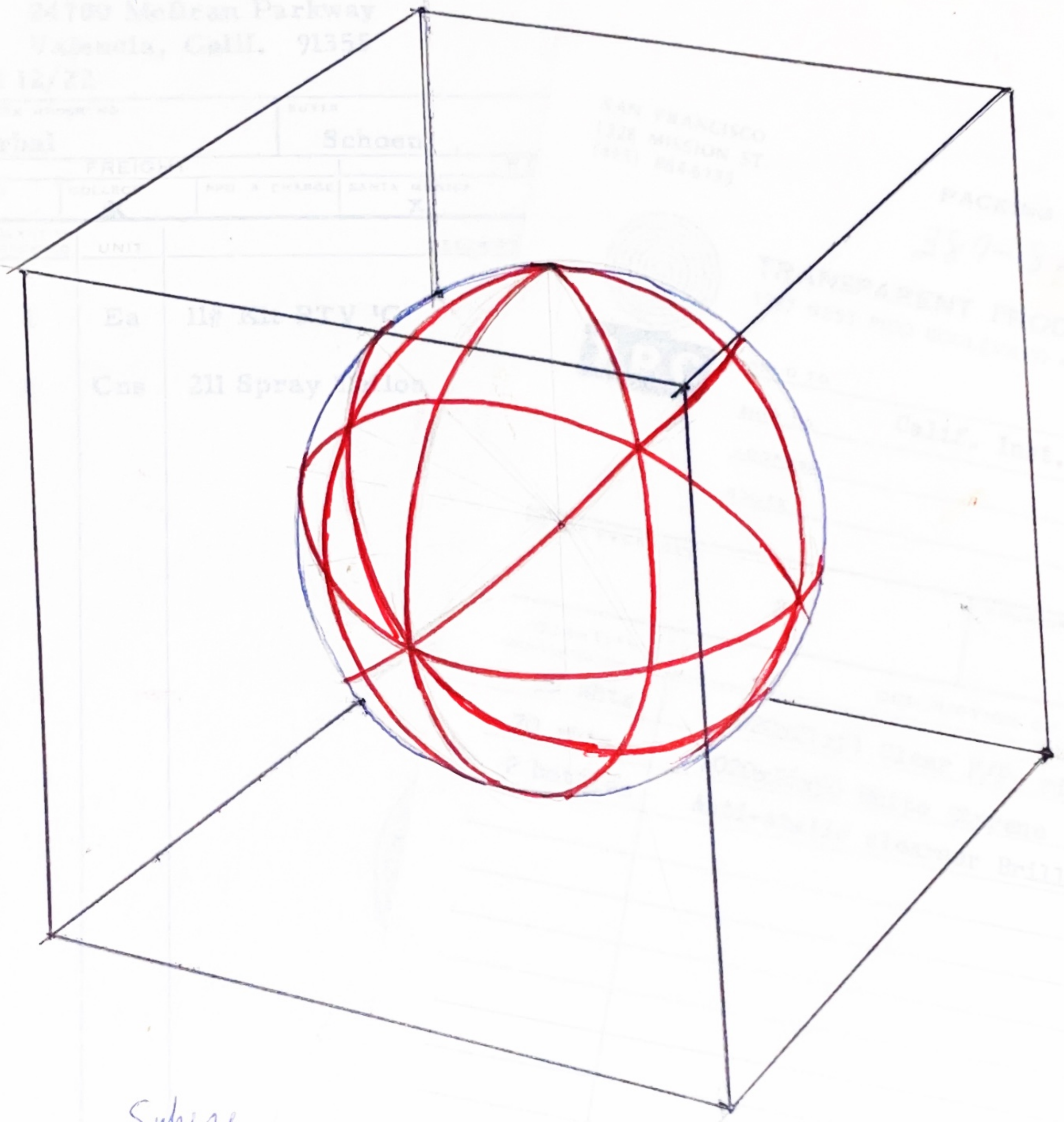
The said soap film which spans a closed loop of wire when ~~it is~~ ~~withdrawn~~ f. the loop is pulled out of a ~~sol~~ soap and water solution is called a minimal surface. ~~This~~ This surface, which ~~has~~ is usually of smaller ~~less~~ area than any other surface bounded by the same loop, is convexo-concave, or saddle-like, at all its points.

Suppose we wish to




30°, 45°, 60°, 90°

final spherical state of P



Sphere

radius = 80% of cube "radius"

Hence, separation of spheres = 1/4 (sphere diam.)

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70 shts	.020x26x50 White Styrene
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		ca	1-3-72

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ITEM	QUANTITY ORDERED	UNIT	DESCRIPTION	CODE	BACK ORD	SHIPPED	UNIT PRICE	TOTAL
1	1	Ea	11# Kit RTV 'G'	70		1		53 90
2	2	Cns	211 Spray Teflon	70		2		3 00
				Sales Tax				2 85
				Shipping				1 10
								60 85

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Form No. 130-1

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TYPICAL PROPERTIES

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Colors	White, Black or Clear
Specific Gravity at 77 F	1.04
Rate of Extrusion (1/8" orifice, 90 psi), grams per min.	500
Flow (sag or slump, 1/8 x 4-inch bead), inches	nil
Cure Characteristics	
(in air, 77 F, 50% R.H.)	
Tack-free Time, minutes	10-20
Cure Time (1/8" thickness), hours	24
Cured	
Durometer Hardness, Shore A Scale	
Tensile Strength, psi	275
Elongation, percent	450
Brittle Point, degrees F	-100
Typical chemical resistance after 3-day cure	
H_2SO_4 (10%)	Excels
Ammonia (50%)	Good
NaOH (20%)	Fair
Oxygen	Excels
Ozone	Excels
Water	Excels

Silastic 732 RTV adhesive/sealant is a product of
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DOW CORNING

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GA = GALLON, LB = POUND, LT = LOT, M = 1000, OZ = OUNCE, QT = QUART, RL = ROLL, SF = SQUARE FOOT

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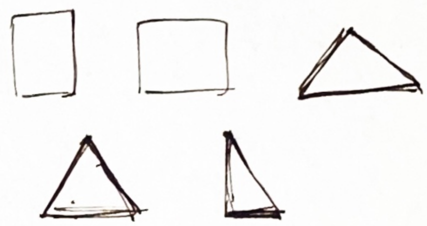
ACKNOWLEDGMENT

Emerson & Cuming, Inc.

The five Platonic solids — or regular polyhedra, ~~or~~ as they are usually called — represent the only possible ways in which identical regular faces (polygons) ~~can be~~ can be joined to form a convex shape whose corners (vertices) are ~~all~~ alike. (A regular ~~face is a poly~~ polygon is one whose edges are all equal and whose face angles are all equal.)

1. Reg. polyhedra — incl. 4, 6, 8, 12, 20
   define terms — describe symmetry what it means.
   The cube is only space filling reg. poly.

   close packing
   periodic structures



2. Plane Tessellations ~~[illegible]~~
   a) 3 reg. ones
   b) 5 "special" ones
   c) dual networks

rotation - reflection
hinging
unsymmetrical motif.

3. Min. surf: D example — hinging
   P — reflection

2 interpenetrating Labyrinths — illustrate by puzzle joint of P
3-dim. networks ~~duals~~
a) surface networks
b) skeletal networks

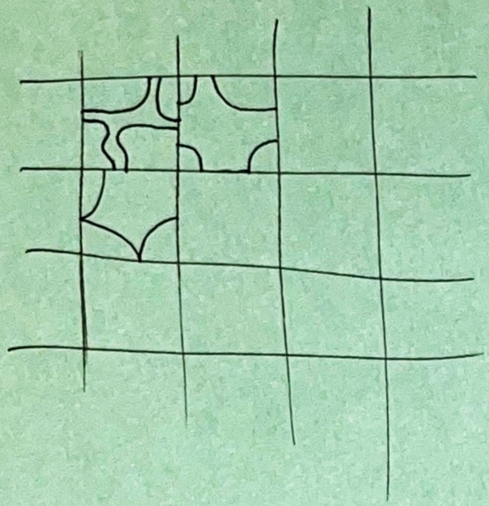
Figs. ~~3~~ 5 and ~~4~~ 6 show the two other examples of spherical tessellations generated by reflection; these display the symmetries of the five Platonic solids, or convex regular polyhedra. The sphere in Fig. ~~3~~ 5 is partitioned into 48 triangles; ~~and~~ that one in Fig ~~4~~ 6, 120 triangles. The symmetries of the tetrahedron, cube, and octahedron (shown in Figs. 7, 8, and 9, respectively) ~~correspond to~~ are based on those of Fig. 5, while the symmetries of the dodecahedron and icosahedron (shown in Figs. 10 and 11, respectively) correspond to those of Fig. 6. Note that the tetrahedron in Fig. 7 is subdivided into only 24 triangles, not 48; the tetrahedron ~~because it~~ has only half as many ~~axes~~ symmetries (24) as the cube or octahedron (48).

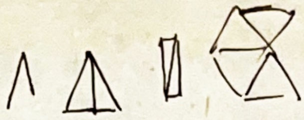
III. TESSELLATIONS OF MINIMAL SURFACES

Some ~~A third~~ Minimal surfaces ~~can~~ can ~~also~~ be symmetrically ~~tessellated by~~ covered by identical ~~A third type of surface which can be~~ polygons. ~~is a special class of~~ ~~In this case~~ In the simplest cases, these ~~polygons~~ are skew polygons bounded by straight line segments; ~~and~~ ~~each~~ every polygon is related to each of its nearest neighbors by rotation through a half-turn about their common edge.

(The plane ~~itself is an example~~ is a special case of a minimal surface; the examples already described in of plane tessellations, the elementary polygons

→ ~~Except for the~~ A simple ~~polygon~~ closed curve spanned by a soap film is a model of a minimal surface, which is a surface of least area in the following sense: ~~that~~ if a sufficiently small closed curve is inscribed anywhere in the surface, ~~then~~ the ~~patch~~ patch of ~~the~~ surface enclosed by the curve has a smaller area than ~~the~~ any other ~~of all~~ of a surface bounded by the same curve. ~~It~~ The plane is the simplest ~~obviously~~ example of ~~an example of~~ a minimal surface~~s, and hence the polygons shown in Fig. 1 are examples of modular units of minimal surface related by reflection or rotation~~. All other ~~minimal surfaces~~ examples minimal surfaces are ~~curved~~ curved, ~~surfaces~~, the curvature being saddle-like at every point of the surface.





~~all~~ Among

~~If we examine~~ the ways in which ~~we can cover~~ ~~among them~~

can be covered / polygons, ~~tiles~~

a plane surface, with identical ~~polygons~~, ~~we find that~~ there

which have ~~based on~~ ~~reflection symmetry~~: / are five which exhibit the simplest kind of reflection

are ~~exactly~~ five ~~ways~~ ~~among them~~ ~~which have a special~~ symmetry:

every / ~~polygon~~ pair of ~~adjacent~~ polygons which share a common edge

~~kind of symmetry~~: ~~each~~ ~~tile~~ ~~is a mirror image~~ are

~~adjacent tile~~ polygon, / mirror

~~of each one next to it~~, with respect to a mirror images of each

standing ~~a~~ along their ~~edge~~ common edge. ~~We will call~~ other,

~~the tiles polygons~~, and we will say that each tile

is reflected into the one next to it.

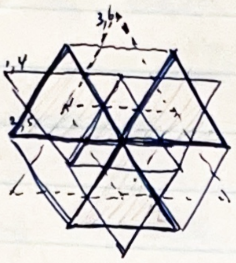
An alternative way to describe such a tessellation (a covering of the infinite plane by polygons fitted together) is to say that each tile is rotated through a half-turn ~~about~~ about each of its edges so as to fall exactly in register on top of a neighboring tile.

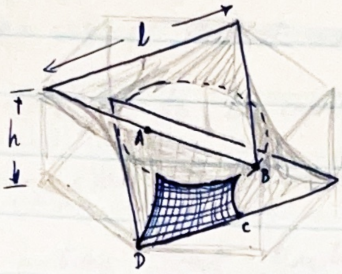
every ~~each~~ ~~every~~ polygon is a mirror image of each of its nearest neighbors ~~neighbor~~.

Among the ways in which a plane surface can be covered with identical polygons, there are five which exhibit the simplest form of reflection symmetry: ~~each po~~ ~~every polygon is a mirror image of each of its neighbors, with respect to~~ every polygon is the ~~mirror im~~ mirror image of each of its nearest neighbors, ~~its neighbors~~, ~~with respect to a mirror plane~~ the "mirror" being the ~~common~~ line lying along their ~~being~~ common edge. ~~Each~~ We can describe such a plane tessellation as being generated by reflection ~~of each polygon in its edges~~. ~~An a~~

~~An~~ is An alternative way of viewing such a ~~p~~ tessellation ~~is to~~ ~~which~~ makes use of the ~~requires that~~ fact that the plane is embedded in 3-dimensional space: ~~each polygon is~~ every ~~each~~ polygon is related to each of its nearest neighbors by rotation through a half-turn about ~~each~~ their common edge.

S: surface based on star-of-David ring.





For D, $\frac{h}{l} = \frac{\sqrt{6}}{6} \cong .408$

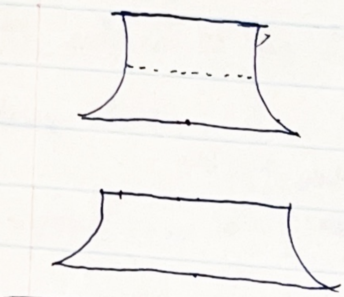
For P, $\frac{h}{l} = \frac{\sqrt{6}}{12} = .204$

When the 2 triangles are separated by a suitable distance, $|AB| = |BC| = |CD| = |DA|$.

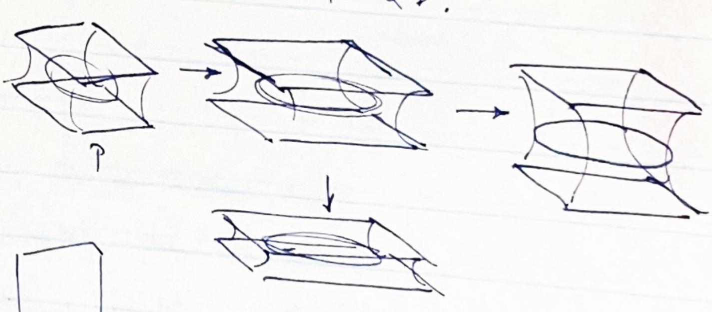
$\boxed{\text{Then } \mathcal{A}(S) \equiv S}$

What is the associate surface which is a balanced linear combination of S & $\mathcal{A}(S)$?

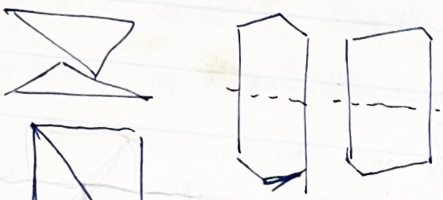
Is it a regular surface? (Probably not!)

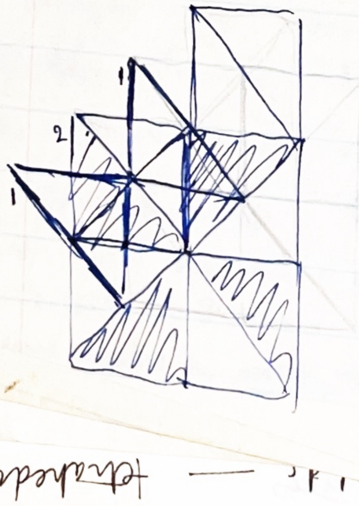


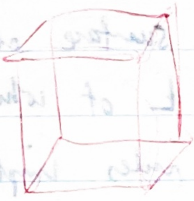
Distorted versions of P & D.

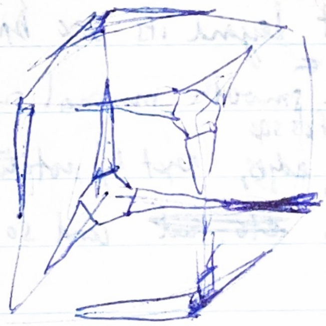


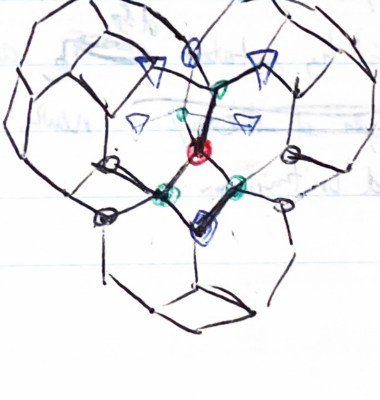
D, P











Schwarz's reflection principle:

Schwarz showed that any straight line L lying in a minimal surface S is an axis of 2-fold rotational symmetry of S. This means that if S is rotated 180° about L, it is mapped onto itself, with its two sides ("front" and "back") interchanged.

Fig. 1 shows ~~an six~~ a minimal surface S_D, which is bounded by a regular skew hexagon inscribed in a cube. S_D has three 2-fold symmetry axes — L_1L_1', L_2L_2', and L_3L_3' — intersecting at its center. In Fig 1b, S_D has been rotated 180° about L_2L_2', with respect to the orientation of S_D shown in Fig. 1a.

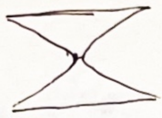
If a finite line segment l lies in a minimal surface S, then the complete minimal surface of which S is a part contains an infinite line L of which l is a segment. ~~In the case of S_D (Fig. 1)~~ (cf. Fig. 1), These rules imply that S_D can be extended without limit beyond its six boundary edges ~~first~~ to generate a single smooth minimal surface, first by rotation of S_D about each of its six edges, next by rotation of the enlarged surface about the free edges ~~edges of the six~~ newly generated, ~~etc. ad~~ and so on, ad infinitum.

~~Th~~ Fig. 1 ~~s~~ shows ~~s~~ a piece of ~~a~~

~~The~~ soap film surface — or minimal surface — ~~shown in Fig. 1~~, which can be extended smoothly beyond its boundaries without limit ~~without ever intersecting itself~~. ~~There is~~ The surface of Fig. 1, ~~This surface~~, which was discovered and analyzed by the mathematician H. A. Schwarz in 1866, is called Schwarz's primitive surface, or simply P.

It is possible ~~probably~~ that there ~~exists~~ is only a finite number of distinct minimal surfaces with this property.) ~~Examples of this general type, type shown in Fig. 1,~~ Minimal surfaces like P, which in their complete form ~~are~~ suggest ~~hollow tubular~~ infinite space frames assembled from hollow tubules, are called infinitely-connected periodic minimal surfaces without self-intersections (ICPMS w/o SI). We ~~shall describe~~ will describe several other examples of such surfaces. We will also examine some of their properties and explain methods for generating additional examples.

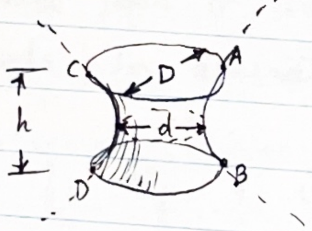
In Fig. 1 it can be seen that ~~there are a number of~~ the straight lines lying in ~~the surface~~ P define ~~form~~ a surface network, and that this surface network divides P into congruent quadrangles, ~~eg~~ e.g., [1234]. Schwarz proved that ~~the~~ if there is a ~~straight~~ finite line segment lying in ~~any~~ a minimal surface, it must be part of ~~an~~ ~~an [infinite]~~ ~~a line~~ ~~an [infinite] line~~. an infinite line. He proved also that any line lying in a minimal surface is an axis of 2-fold rotational symmetry of the surface.



Before considering the difficult topic of ICPMS, ~~let us~~ we will examine some simpler examples of minimal surface soap films.

First, ~~suppose~~ let us ~~assume~~ recall the definition of a minimal surface: ~~it is a surface having the property that if you draw a suf~~ it is a surface of least area, in the sense that if you draw a sufficiently small simple closed curve anywhere on the surface, the portion of the surface contained within the closed curve has less area than any other surface bounded by that closed curve.

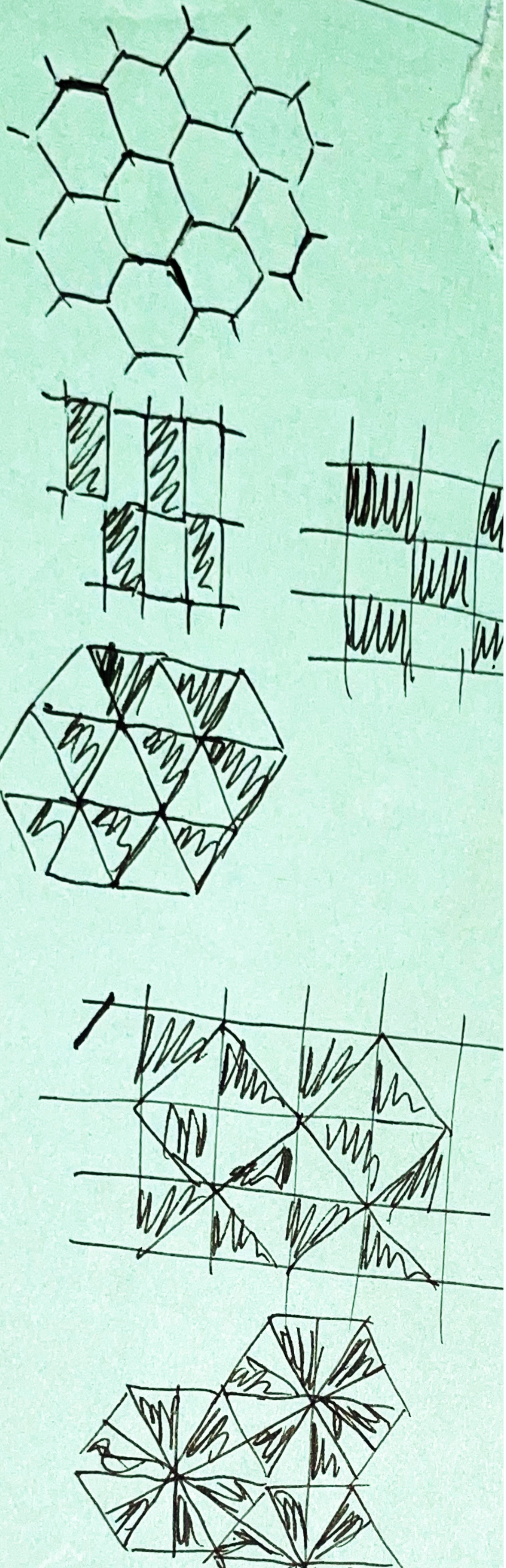
Obviously, the ~~plane~~ infinite plane is an example of a minimal surface. It is the only "easy" example. ~~Perhaps~~ Probably the next most "simple" ~~next~~ simplest of example of a ~~all other~~ minimal surfaces is the catenoid, which can be modelled by a soap film suspended between two coaxial circular rings. The catenoid owes its name to the fact that any of its meridional curves side ~~profile curves~~, ~~of a catenoid~~ e.g., ~~the curves~~ AB & CD, has the form of ~~is~~ a catenary,



which is the shape assumed by a uniform ~~perfectly~~ flexible cable hanging ~~under the influence of its own~~ under the influence of its own weight between supports at its ends.

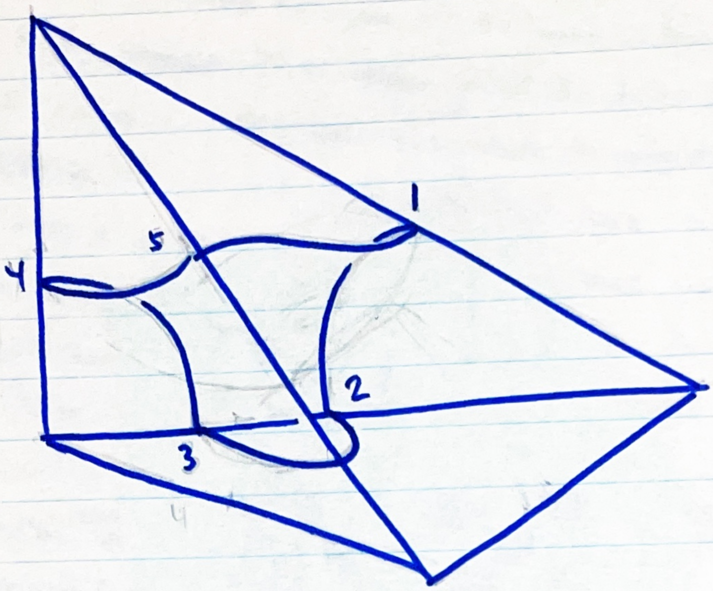
~~The~~ When loops of wire ~~loops~~ are dipped into soap solution to make a ~~soap film model of the~~ catenoid, the limiting height of the resulting soap film (h in Fig.) is determined by the diameter D of the rings. As h approaches its maximum value, the diameter d of the ~~of the waist of the "hole"~~ "neck" in the equatorial plane of the film becomes smaller and smaller, and eventually vanishes.

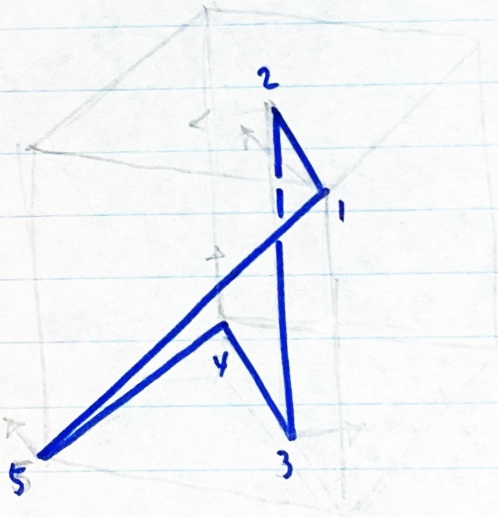
Rule



which

Fig. 1 ~~symmetry operation~~


1
5
4
2
3


2
1
4
5
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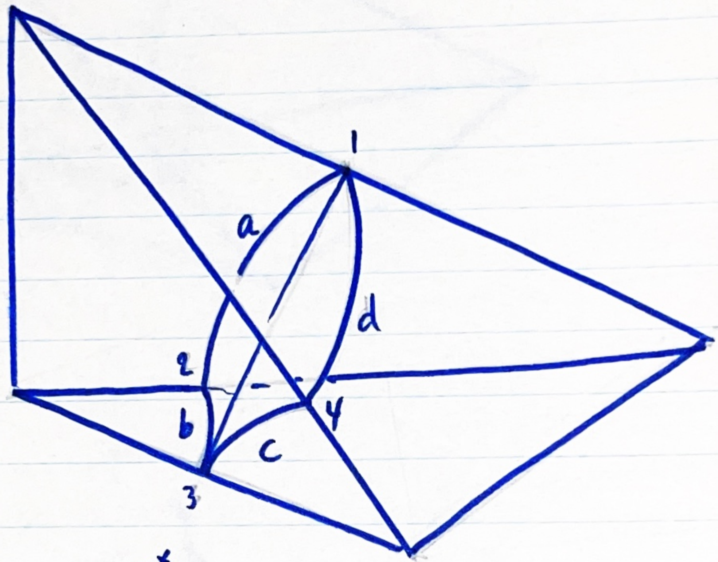
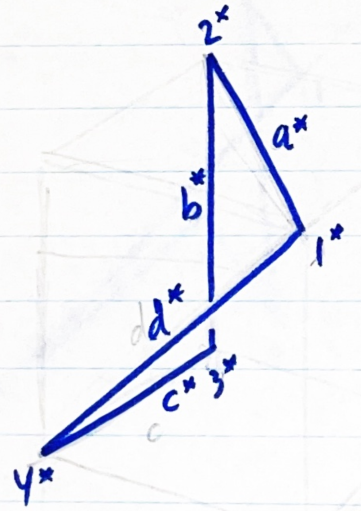
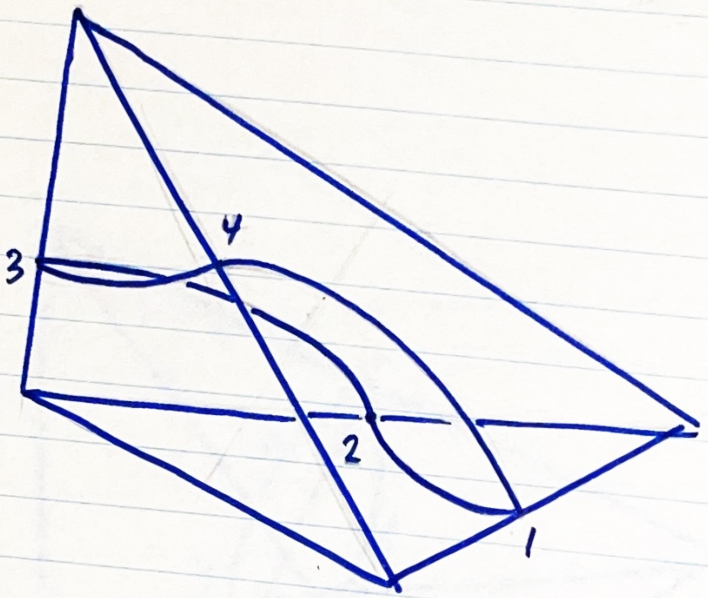
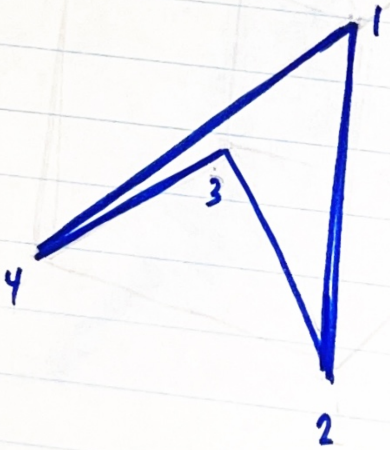
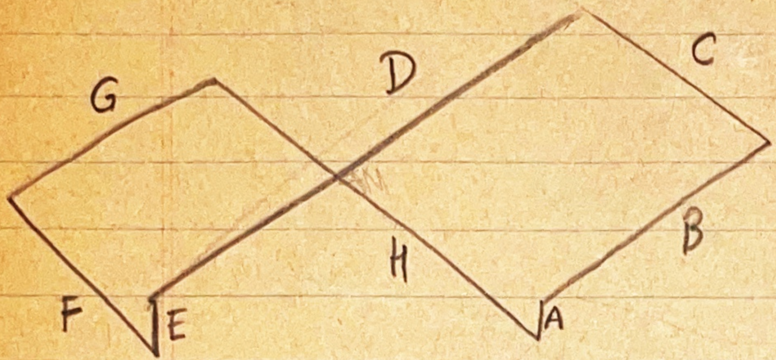

1
a
d
2
b
4
c
3

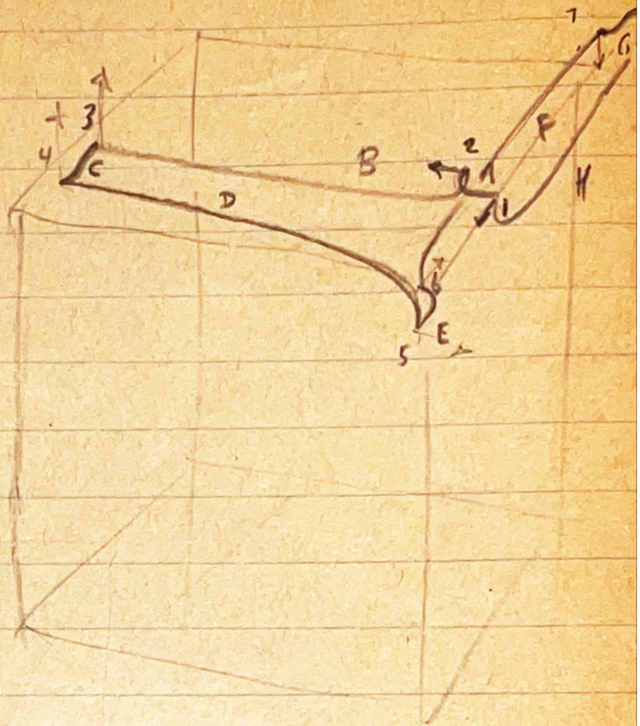

2*
a*
b*
1*
d*
c*
3*
4*


4
3
2
1

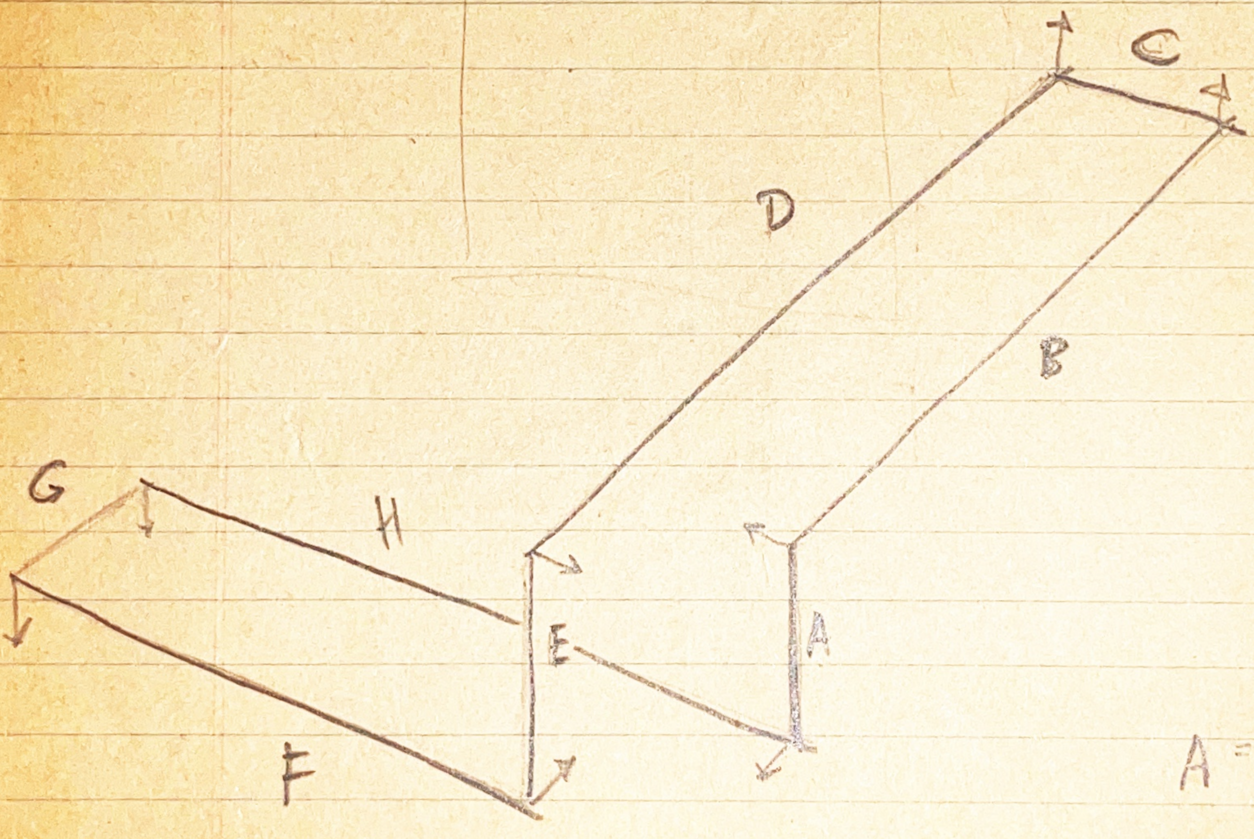

1
3
4
2

Require that $\left\{\begin{array}{l} B \,\&\, G \text{ are coplanar} \\ C \,\&\, F \text{ are coplanar} \end{array}\right.$





Initially,



$A = E$

CHECKERBOARD PATTERNS GENERATED BY REFLECTION (a generalization of the ordinary square checkerboard)

It is common knowledge that identical flat tiles of suitable shape can be fitted together to cover a plane surface so as to make a symmetrical pattern. The most symmetrical examples of such tile shapes are the equilateral triangle, the square, and the regular hexagon, but there is no limit to the number of ~~choices for the~~ ~~other~~ possibilities for the shape~~s~~ of a plane polygon~~s~~ which will fit together with replicas of ~~[illegible]~~ itself, without overlapping, to cover a plane surface symmetrically. Such a covering of the infinite plane is called a unary ~~plane tessellation~~ plane tessellation.

We are interested in a ~~particular kind~~ restricted type of unary plane tessellation, which we will call a checkerboard generated by reflection. We define such a pattern by the following rules:

1) ~~Each~~ ~~separate~~ every ~~[illegible]~~ tile is an exact ~~replica~~ duplicate of a ~~finite~~ tile, which is colored black on one side and white on the other, and whose boundary is a finite ~~[illegible]~~ straight-edged plane polygon~~s~~, not necessarily convex;